# The Higgs transverse momentum at $N^3LL'$+NNLO with RadISH

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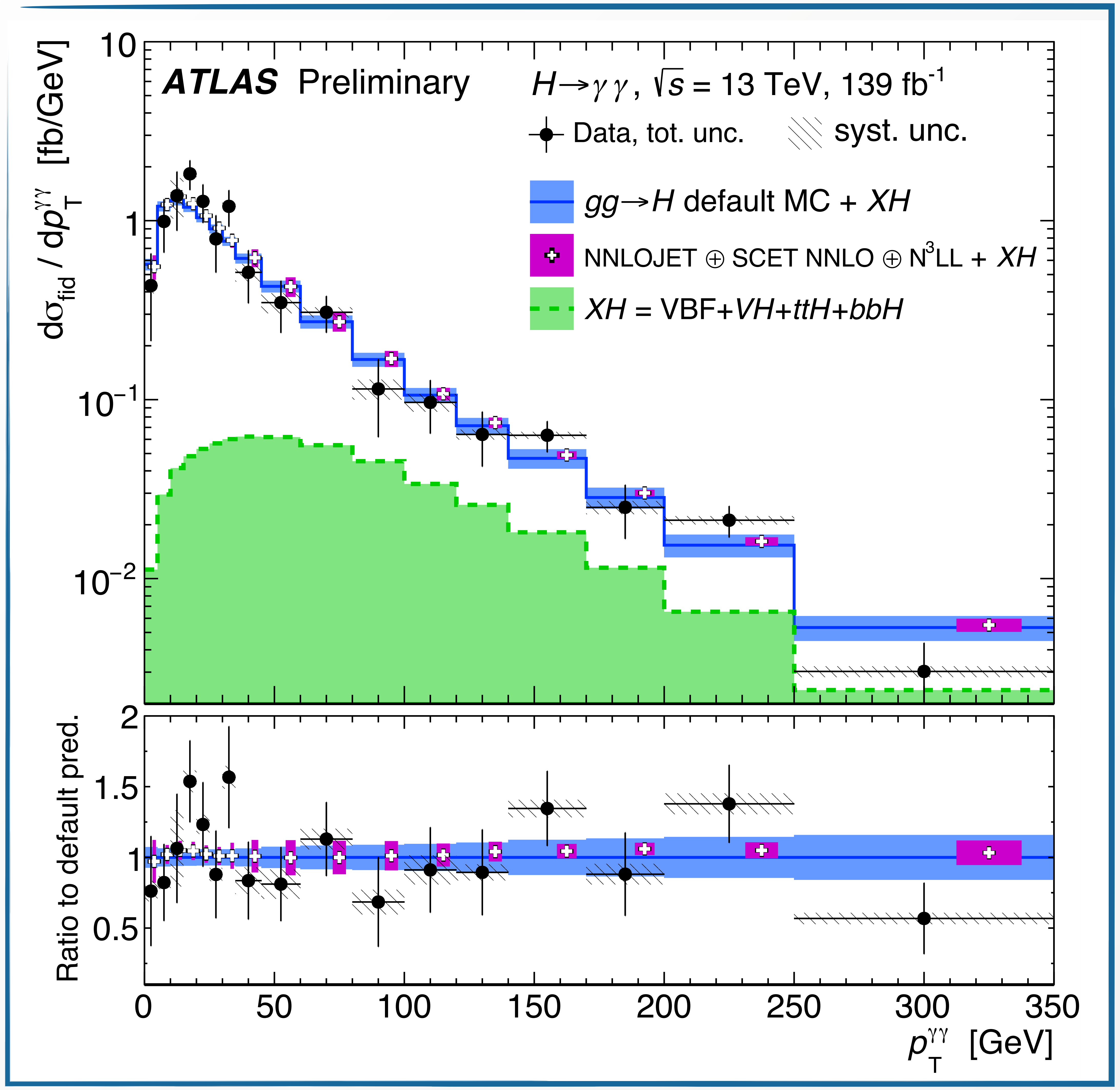
FNSNF Swiss National Science Foundation

*Based on: Re, LR, Torrielli 2104.07509*

*RadISH formalism: Monni, Re, Torrielli 1604.02191, Bizon, Monni, Re, LR, Torrielli 1705.09127*

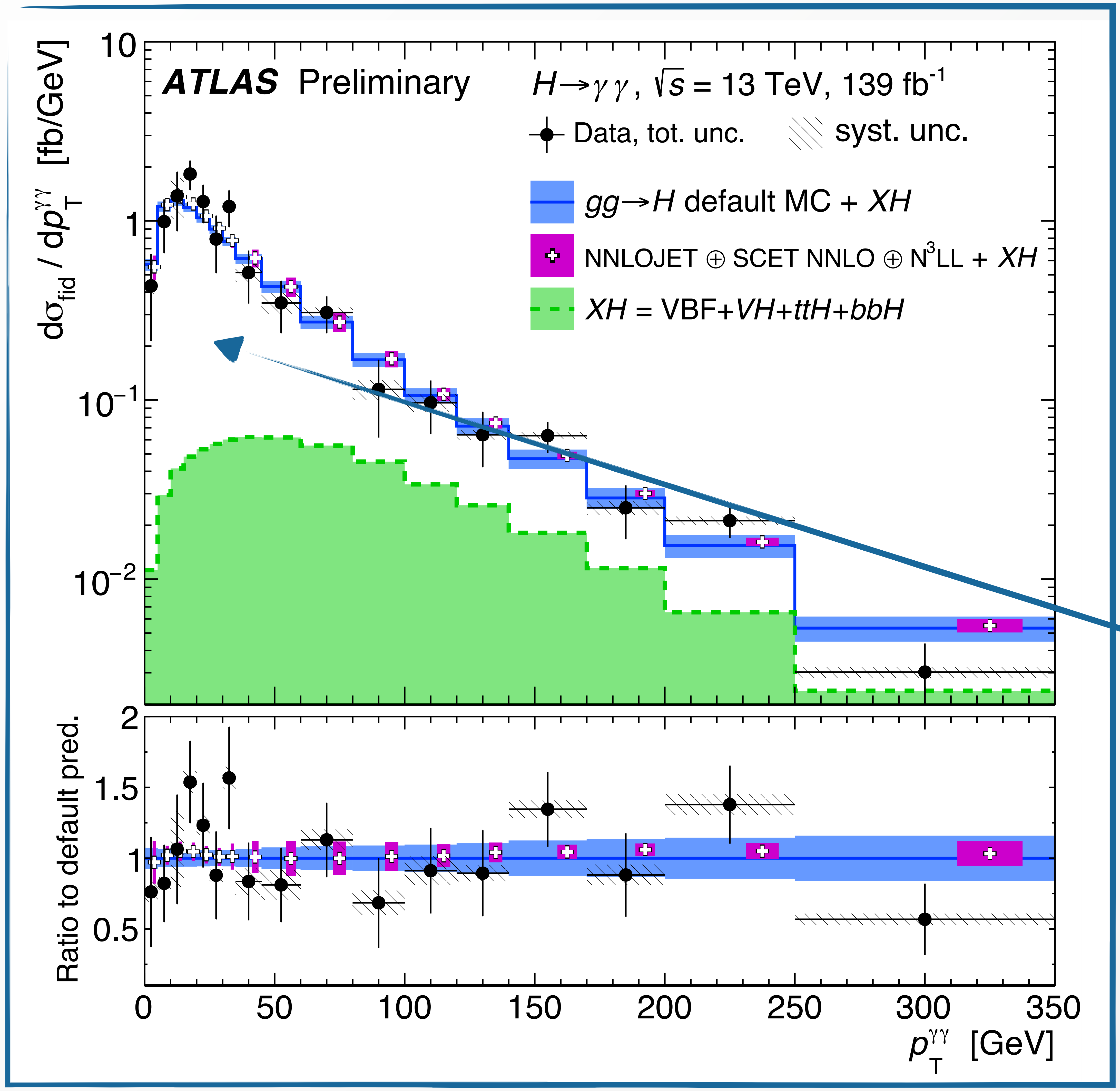
# The Higgs transverse momentum




[ATLAS-CONF-2019-029]

- Relatively easy to measure
- Sensitivity to New Physics (e.g. **light Yukawa** couplings, **trilinear** Higgs coupling) [Bishara *et al.* '16][Soreq *et al.* '16] [Bizon *et al.* 1610.05771]
- Experimental analyses categorize events into **jet bins** according to the jet multiplicity
- Increased sensitivity to Higgs boson kinematics, spin-CP properties, BSM effects…

# The Higgs transverse momentum




[ATLAS-CONF-2019-029]

Large **transverse momentum** logarithms

$$L = \ln(p_t^H/m_H) \qquad p_t^H \ll m_H$$

# The Higgs transverse momentum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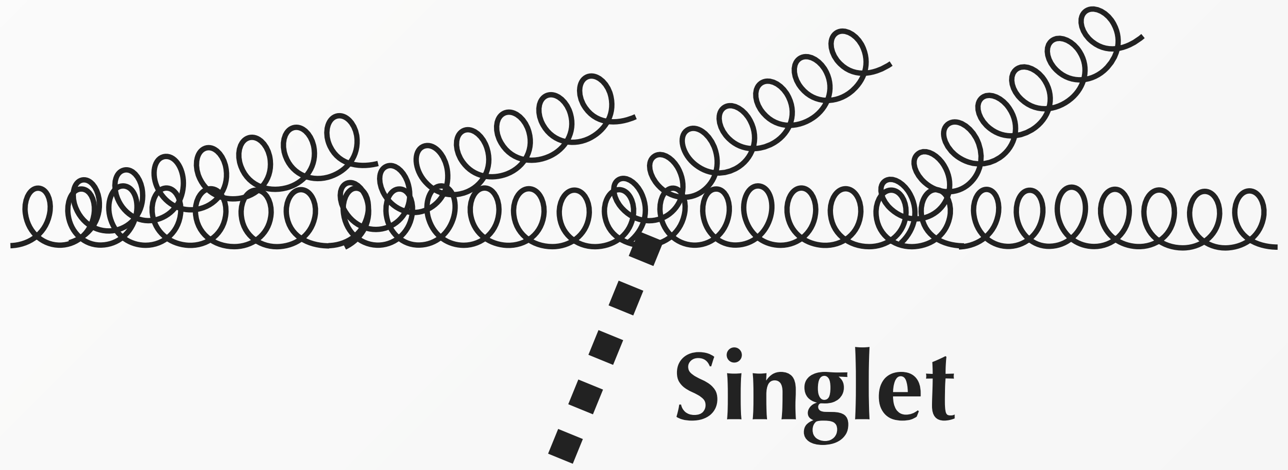
Fixed order predictions no longer reliable:
**all order resummation** of the perturbative series mandatory

# Resummation of the transverse momentum spectrum

Resummation of transverse momentum is delicate because $p_t$ is a **vectorial quantity**

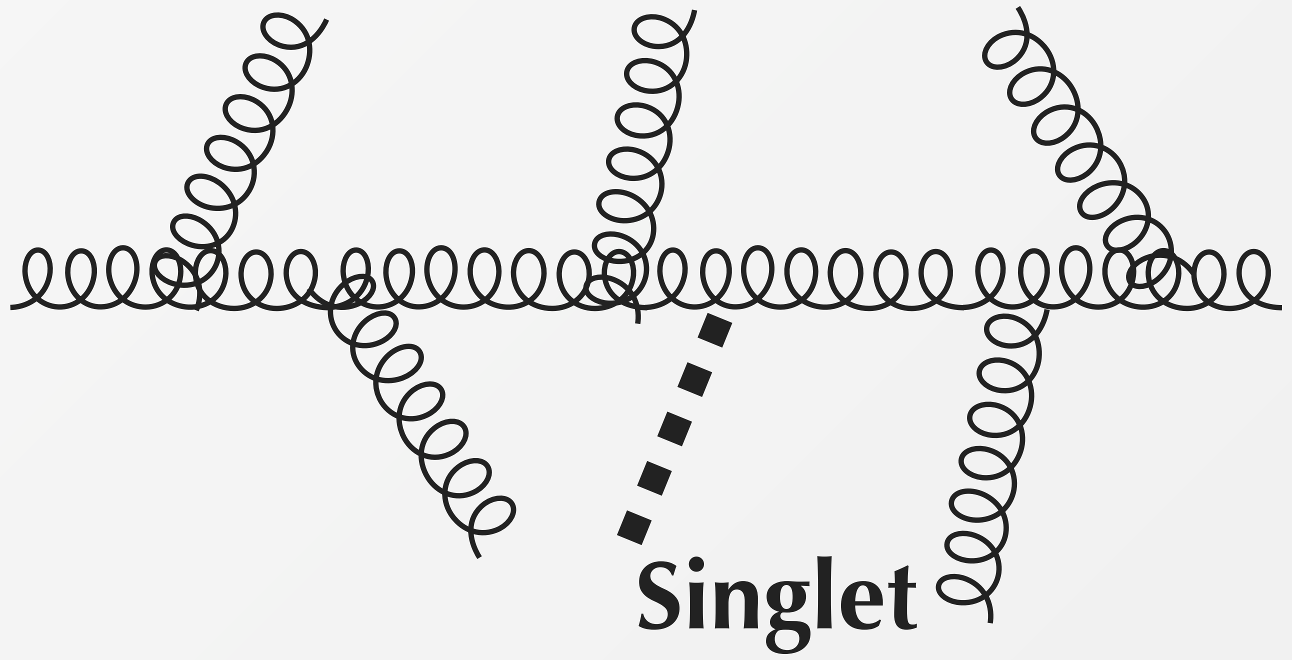
**Two concurring mechanisms** leading to a system with small $p_t$




$$p_t^2 \sim k_{t,i}^2 \ll m_H^2$$

cross section naturally suppressed as there is no phase space left for gluon emission (Sudakov limit)

**Exponential suppression**




$$\sum_{i=1}^{n} \vec{k}_{t,i} \simeq 0$$

**Large kinematic cancellations**

$p_t$ ~0 far from the Sudakov limit

**Power suppression**

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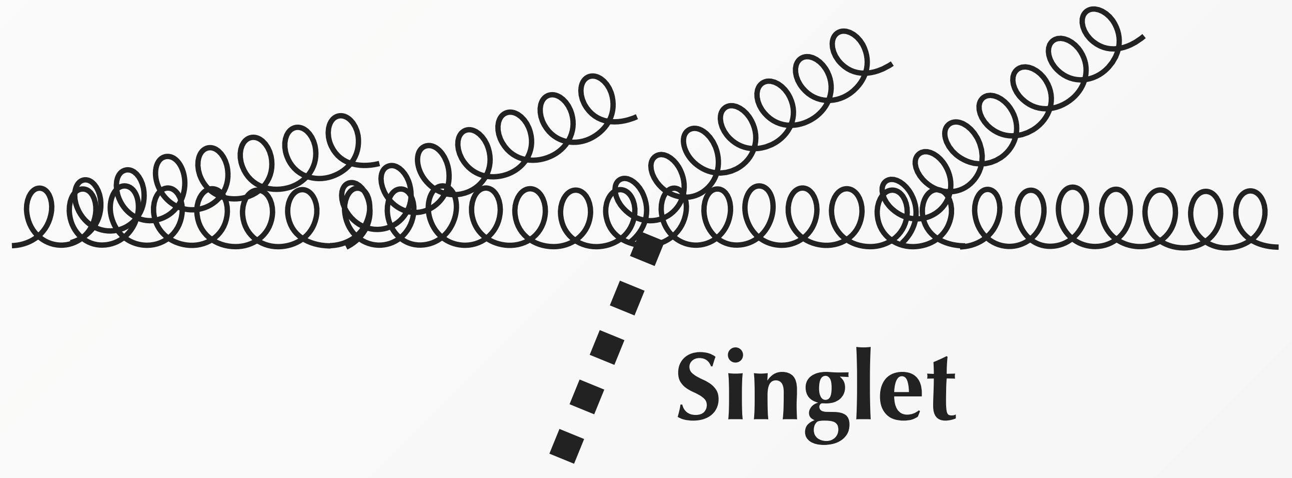


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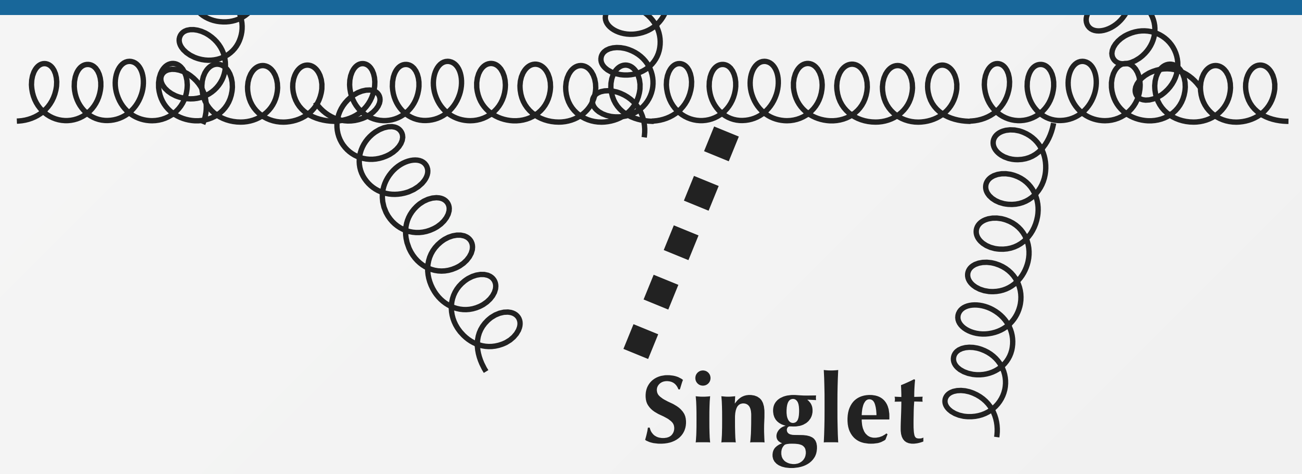
## Dominant at small $p_t$

[Parisi, Petronzio, 1979]

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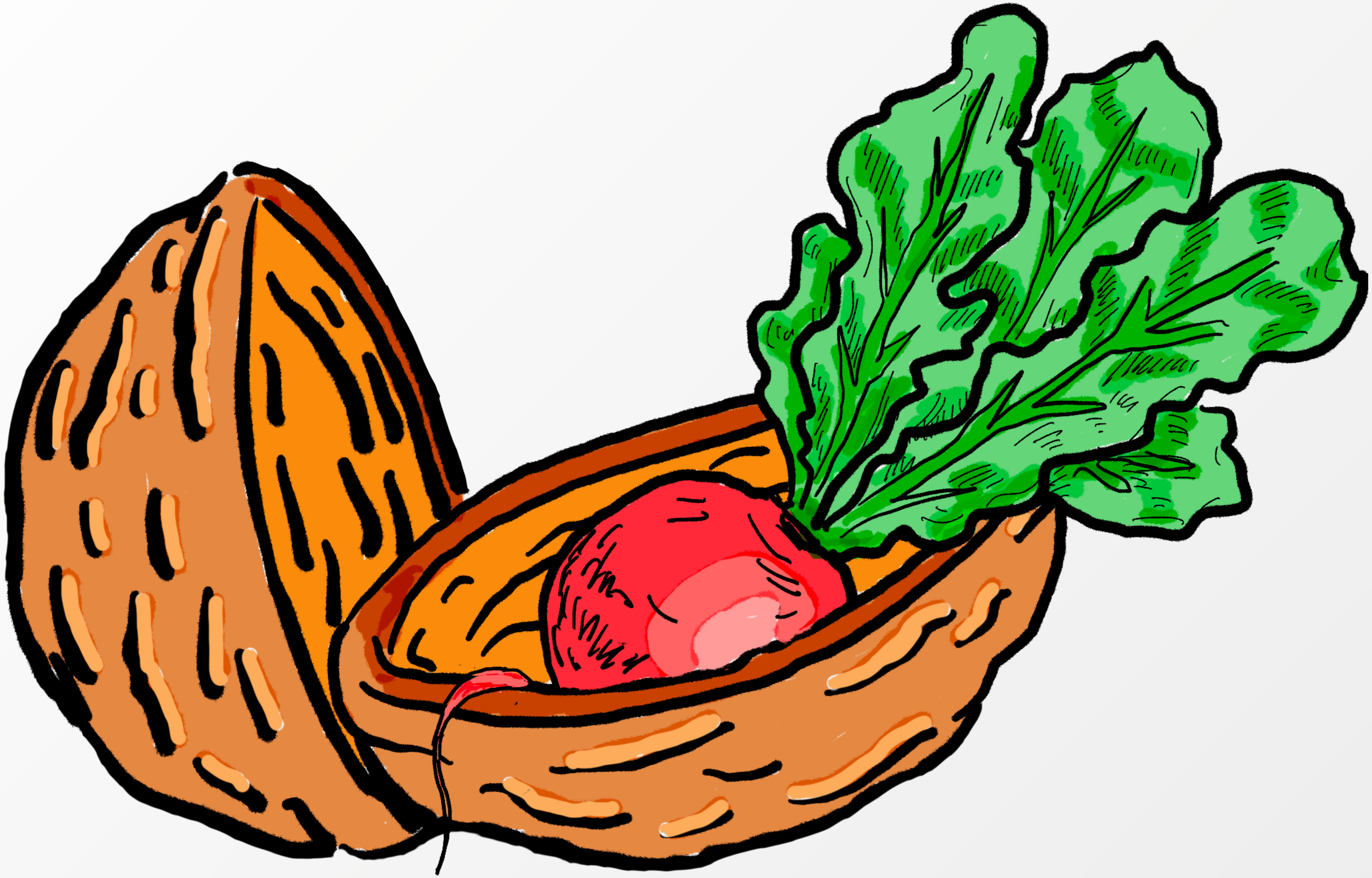
# RadISH in a nutshell



# RadISH in a nutshell

Resummation of the $p_t$ spectrum in direct space

Result at NLL accuracy (with fixed PDFs) can be written as

$$\sigma(p_t) = \sigma_0 \int \frac{dv_1}{v_1} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(v_1)}$$

**Unresolved**

$$v_i = k_{t,i}/m_H, \quad \zeta_i = v_i/v_1$$

$$\times \, \epsilon^{R'(v_1)} R'(v_1) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(\zeta_i v_1) \, \Theta\left(p_t - |\vec{k}_{t,i} + \cdots \vec{k}_{t,n+1}|\right)$$

**Resolved**

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**Resolved**

**Sudakov** and **azimuthal mechanisms** accounted for, **no assumption** on $k_{t,i}$ vs $p_t$ hierarchy.

**Subleading effects retained**: no divergence at small $p_t$, Parisi-Petronzio power-like behaviour respected

**Logarithmic accuracy** defined in terms of $\ln(m_H/k_{t1})$

Result formally equivalent to the $b$-space formulation [Bizon, Monni, Re, LR, Torrielli '17]

# All-order formula in Mellin space [Bizon, Monni, Re, LR, Torrielli '17]

Now include effect of **collinear radiation** and terms beyond NLL accuracy

$$\frac{d\Sigma(v)}{d\Phi_B} = \int_{\mathscr{C}_1} \frac{dN_1}{2\pi i} \int_{\mathscr{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) \mathbf{f}_{N_2}(\mu_0),$$

**Unresolved**

$$\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-\mathbf{R}(\epsilon k_{t1})}$$

$$\times \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$

**Resolved**

$$\times \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right),$$

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$v = p_t/M$

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**Unresolved**

**Hard-virtual coefficient**

$v = p_t/M$

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**Resolved**

$$\times \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$
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# All-order formula in Mellin space [Bizon, Monni, Re, LR, Torrielli '17]

Now include effect of **collinear radiation** and terms beyond NLL accuracy

$$\frac{d\Sigma(v)}{d\Phi_B} = \int_{\mathscr{C}_1} \frac{dN_1}{2\pi i} \int_{\mathscr{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) \mathbf{f}_{N_2}(\mu_0),$$

**Collinear coefficient functions and their RGE**

**Unresolved**

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$$\times \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$

$v = p_t/M$

**Resolved**

$$\times \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$

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**Unresolved**

**DGLAP evolution**

$v = p_t/M$

$$\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-\mathbf{R}(\epsilon k_{t1})}$$

$$\times \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$

**Resolved**

$$\times \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$

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**Unresolved**

**Sudakov radiator**

$$\hat{\boldsymbol{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-\mathbf{R}(\epsilon k_{t1})}$$

$$v = p_t/M$$

$$\times \exp\left\{-\sum_{\ell=1}^{2} \left( \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \boldsymbol{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \boldsymbol{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t)) \right)\right\}$$

**Resolved**

$$\times \sum_{\ell_1=1}^{2} \left( \mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \boldsymbol{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \boldsymbol{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1})) \right)$$

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# Inclusion of $\mathrm{N^3LL'}$ effects in RadISH [Re, LR, Torrielli '21]

Capture **all constant terms** of relative order $\mathcal{O}(\alpha_s^3)$

- $\alpha_s^3$ is $\mathrm{N^4LL}$ (since $\alpha_s^n L^{n-3}$) but sufficient to get all $\alpha_s^n L^{2n-6}$ in the cumulant
- Allows for the computation of **$\mathrm{N^3LO}$ cross section** for H, DY production based on $p_t$-slicing methods

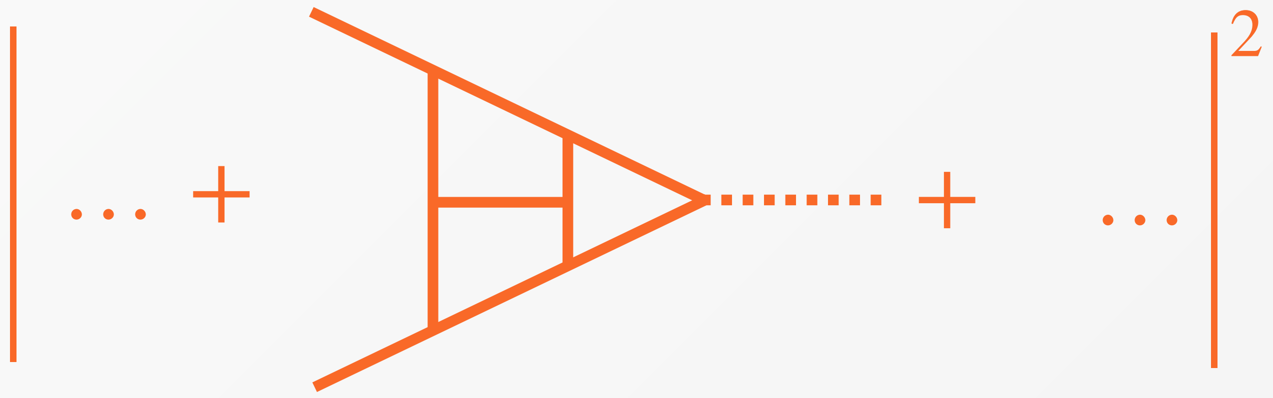
[Billis et al. '21][Cieri et al. '21] [Chen et al. '21]

# Inclusion of N³LL′ effects in RadISH [Re, LR, Torrielli '21]

Sources of N³LL′ correction, neglected in previous RadISH implementation

**Three-loop hard-virtual coefficient**

$$H(\alpha_s) = 1 + \left(\frac{\alpha_s}{2\pi}\right) H_1 + \left(\frac{\alpha_s}{2\pi}\right)^2 H_2 + \left(\frac{\alpha_s}{2\pi}\right)^3 H_3$$




[Gehrmann et al. '10]

Three-loop Wilson coefficient for Higgs EFT

[Schroder, Steinhauser '05]

$$\hat{\Sigma}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right]\int_0^M \frac{dk_{t1}}{k_{t1}}\int_0^{2\pi}\frac{d\phi_1}{2\pi}e^{-\mathbf{R}(\epsilon k_{t1})}$$
$$\times \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$
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Sources of N³LL′ correction, neglected in previous RadISH implementation

**Three-loop coefficient functions**

$$C(\alpha_s, z) = \delta(1-z) + \left(\frac{\alpha_s}{2\pi}\right) C_1(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 C_2(z) + \left(\frac{\alpha_s}{2\pi}\right)^3 C_3(z)$$

[Li, Zhu '16][Vladimirov '16][Luo et al. '19][Ebert et al. '20]

For Higgs production: **two-loop G coefficient functions**

[Catani, Grazzini '11]

$$G(\alpha_s, z) = \left(\frac{\alpha_s}{2\pi}\right) G_1(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 G_2(z)$$

[Luo et al. '19]

$$\hat{\Sigma}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-\mathbf{R}(\epsilon k_{t1})}$$
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**Constants terms coming from the Sudakov**

$$\hat{\Sigma}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right]\int_0^M \frac{dk_{t1}}{k_{t1}}\int_0^{2\pi}\frac{d\phi_1}{2\pi}e^{-\mathbf{R}(\epsilon k_{t1})}$$

$$\times \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$

$$\times \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$

$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=2}^{n+1}\int_{\epsilon}^{1}\frac{d\zeta_i}{\zeta_i}\int_0^{2\pi}\frac{d\phi_i}{2\pi}\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right),$$

$$\times \sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi}\mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right)$$

$$R(k_{t1}) = -\log\frac{M}{k_{t1}}g_1 - g_2 - \left(\frac{\alpha_s}{\pi}\right)g_3 - \left(\frac{\alpha_s}{\pi}\right)^2 g_4 - \left(\frac{\alpha_s}{\pi}\right)^3 g_5$$

Resummation scale $Q \sim M$

$$\ln\frac{M}{k_{t1}} \to \ln\frac{Q}{k_{t1}} + \ln\frac{M}{Q}$$

Constant terms expanded in $\alpha_s$ and included in $H$

# Inclusion of N³LL′ effects in RadISH [Re, LR, Torrielli '21]

Sources of N³LL′ correction, neglected in previous RadISH implementation

## Constants terms coming from resolved contributions

$$\Gamma(\alpha_s) = \Gamma^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)\Gamma^{(1)}$$

$$\Gamma^{(C)}(\alpha_s) = \left(\frac{\alpha_s}{2\pi}\right)^2\Gamma^{(C,1)}$$

$$\hat{\Sigma}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right]\int_0^M \frac{dk_{t1}}{k_{t1}}\int_0^{2\pi}\frac{d\phi_1}{2\pi}e^{-\mathbf{R}(\epsilon k_{t1})}$$
$$\times \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$
$$\times \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$
$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=2}^{n+1}\int_\epsilon^1\frac{d\zeta_i}{\zeta_i}\int_0^{2\pi}\frac{d\phi_i}{2\pi}\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right),$$
$$\times \sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi}\mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right)$$

# Momentum-space formula at N³LL

$$
\begin{aligned}
\frac{d\Sigma(v)}{d\Phi_B} = & \int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}\partial_L\left(-e^{-R(k_{t1})}\mathscr{L}_{\mathrm{N^3LL}}(k_{t1})\right)\int d\mathcal{Z}\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right) \\
& + \int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(k_{t1})}\int d\mathcal{Z}\int_0^1 \frac{d\zeta_s}{\zeta_s}\frac{d\phi_s}{2\pi}\Bigg\{\left(R'(k_{t1})\mathscr{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L\mathscr{L}_{\mathrm{NNLL}}(k_{t1})\right) \\
& \times\left(R''(k_{t1})\ln\frac{1}{\zeta_s} + \frac{1}{2}R'''(k_{t1})\ln^2\frac{1}{\zeta_s}\right) - R'(k_{t1})\left(\partial_L\mathscr{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi}\alpha_s^2(k_{t1})\hat{P}^{(0)}\otimes\mathscr{L}_{\mathrm{NLL}}(k_{t1})\ln\frac{1}{\zeta_s}\right) \\
& + \frac{\alpha_s^2(k_{t1})}{\pi^2}\hat{P}^{(0)}\otimes\hat{P}^{(0)}\otimes\mathscr{L}_{\mathrm{NLL}}(k_{t1})\Bigg\} \\
& \times\left\{\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_s)\right) - \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)\right\} \\
& + \frac{1}{2}\int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(k_{t1})}\int d\mathcal{Z}\int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}}\frac{d\phi_{s1}}{2\pi}\int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}}\frac{d\phi_{s2}}{2\pi}R'(k_{t1})\Bigg\{\mathscr{L}_{\mathrm{NLL}}(k_{t1})\left(R''(k_{t1})\right)^2\ln\frac{1}{\zeta_{s1}}\ln\frac{1}{\zeta_{s2}} - \partial_L\mathscr{L}_{\mathrm{NLL}}(k_{t1})R''(k_{t1})\left(\ln\frac{1}{\zeta_{s1}} + \ln\frac{1}{\zeta_{s2}}\right) \\
& + \frac{\alpha_s^2(k_{t1})}{\pi^2}\hat{P}^{(0)}\otimes\hat{P}^{(0)}\otimes\mathscr{L}_{\mathrm{NLL}}(k_{t1})\Bigg\} \\
& \times\Big\{\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s1},k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s1})\right) - \\
& \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)\Big\} + \mathcal{O}\left(\alpha_s^n\ln^{2n-6}\frac{1}{v}\right).
\end{aligned}
$$

# Momentum-space formula at N³LL′

$$
\begin{aligned}
\frac{d\Sigma(v)}{d\Phi_B} =& \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathscr{L}_{\mathrm{N^3LL'}}(k_{t1})\right) \int d\mathcal{Z} \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \\
&+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z} \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left( R'(k_{t1}) \mathscr{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathscr{L}_{\mathrm{NNLL}}(k_{t1}) \right) \\
&\times \left( R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left( \partial_L \mathscr{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathscr{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s} \right) \\
&+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathscr{L}_{\mathrm{NLL'}}(k_{t1}) - \beta_0 \frac{\alpha_s^3(k_{t1})}{\pi^2} \left( \hat{P}^{(0)} \otimes \hat{C}^{(1)} + \hat{C}^{(1)} \otimes \hat{P}^{(0)} \right) \otimes \mathscr{L}_{\mathrm{NLL}}(k_{(t1)}) + \frac{\alpha_s^3(k_{t1})}{\pi^2} 2\beta_0 \ln \frac{1}{\zeta_s} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathscr{L}_{\mathrm{NLL}}(k_{(t1)}) \\
&+ \frac{\alpha_s^3(k_{t1})}{2\pi^2} \left( \hat{P}^{(0)} \otimes \hat{P}^{(1)} + \hat{P}^{(1)} \otimes \hat{P}^{(0)} \right) \otimes \mathscr{L}_{\mathrm{NLL}}(k_{(t1)}) \Bigg\} \times \left\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \right\} \\
&+ \frac{1}{2} \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z} \int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}} \frac{d\phi_{s1}}{2\pi} \int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}} \frac{d\phi_{s2}}{2\pi} R'(k_{t1}) \Bigg\{ \mathscr{L}_{\mathrm{NLL}}(k_{t1}) \left(R''(k_{t1})\right)^2 \ln \frac{1}{\zeta_{s1}} \ln \frac{1}{\zeta_{s2}} - \partial_L \mathscr{L}_{\mathrm{NLL}}(k_{t1}) R''(k_{t1}) \left( \ln \frac{1}{\zeta_{s1}} + \ln \frac{1}{\zeta_{s2}} \right) \\
&+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathscr{L}_{\mathrm{NLL}}(k_{t1}) + \frac{\alpha_s^2(k_{t1})}{\pi^2} \left( \ln \frac{1}{\zeta_{s1}} + \ln \frac{1}{\zeta_{s2}} \right) R''(k_{t1}) \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathscr{L}_{\mathrm{NLL}}(k_{(t1)}) - \ln \frac{1}{\zeta_{s1}} \ln \frac{1}{\zeta_{s2}} (R''(k_{t1})^2 \partial_L \mathscr{L}_{\mathrm{NLL}}(k_{(t1)}) \\
&+ \frac{\alpha_s^2(k_{t1})}{\pi^3} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathscr{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \times \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1}, k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1})\right) - \\
&\Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\} + \mathcal{O}\left( \alpha_s^n \ln^{2n-7} \frac{1}{v} \right)
\end{aligned}
$$

Luminosity factors now contains $H_3$ and $C_3$

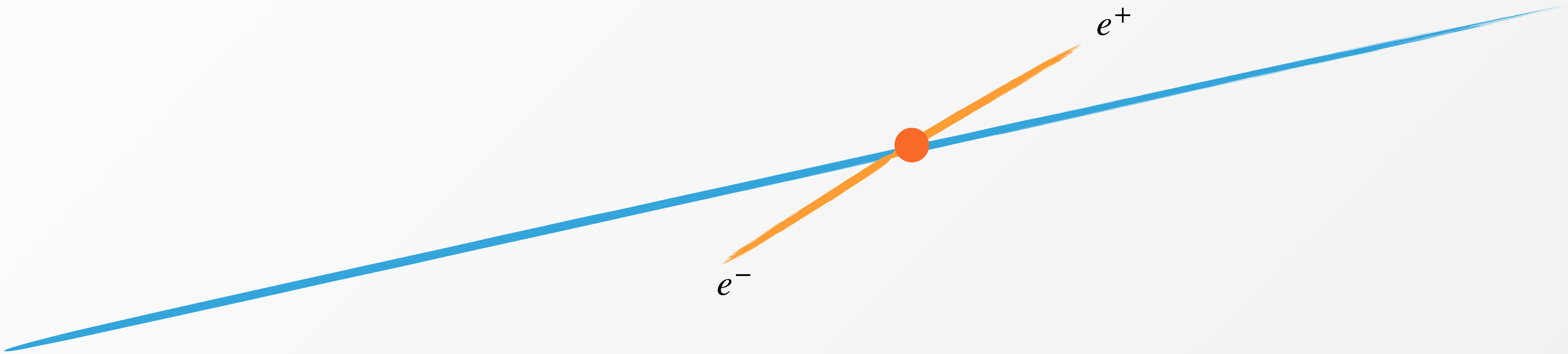
**New structures appearing at $\alpha_s^3$**

Convolution structure obtained after Mellin inversion

Extra column of logs predicted

# Inclusion of transverse recoil effects

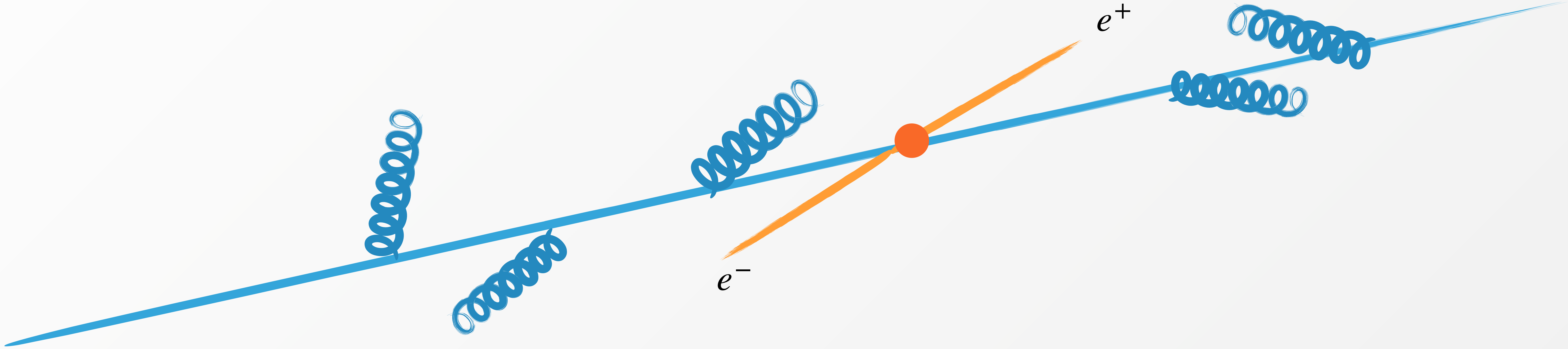
[Catani et al '15]




Born matrix element
evaluated at $p_t = 0$

# Inclusion of transverse recoil effects

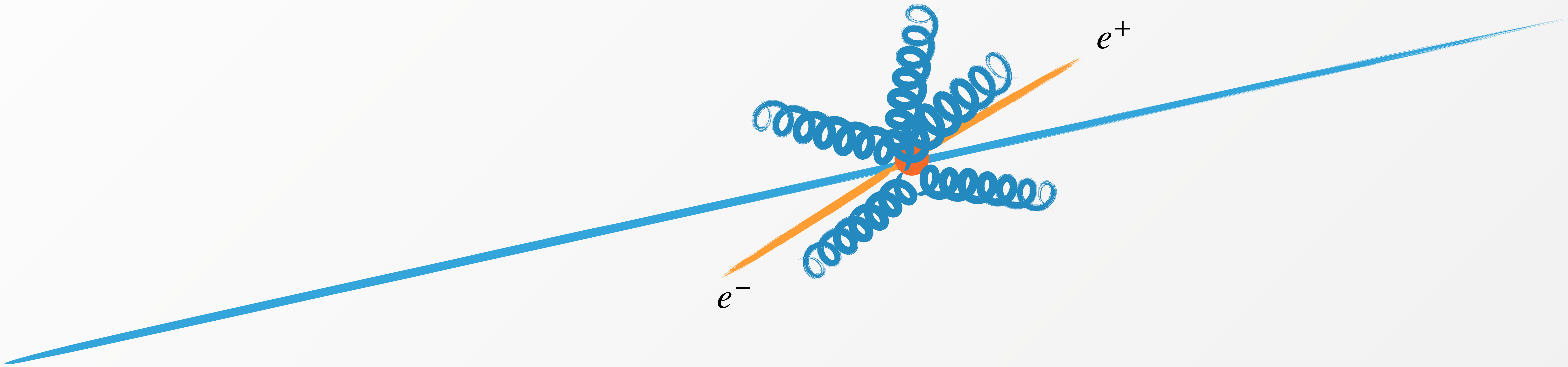
[Catani et al '15]




Generate singlet $p_t$ by QCD radiation

# Inclusion of transverse recoil effects

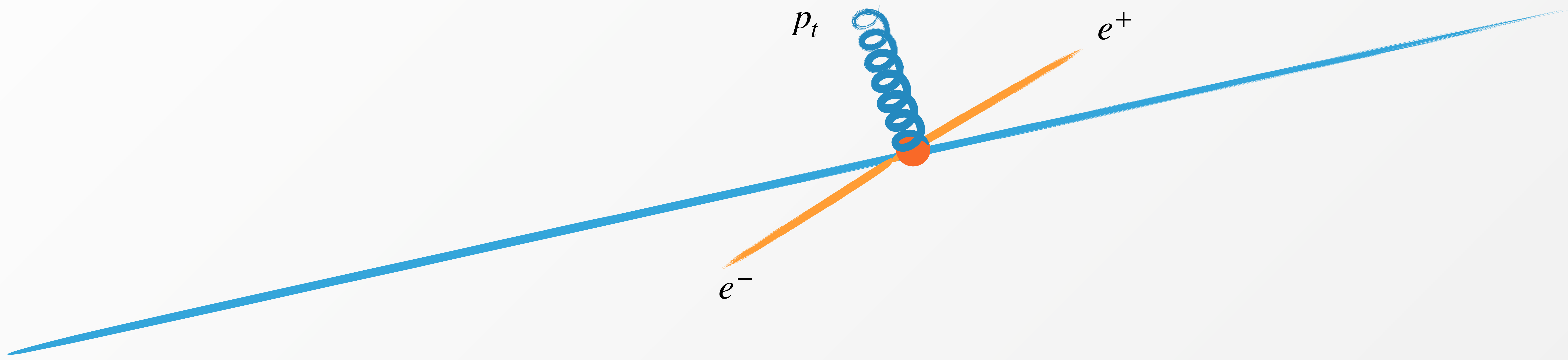
[Catani et al '15]




Generate singlet $p_t$ by QCD radiation

# Inclusion of transverse recoil effects

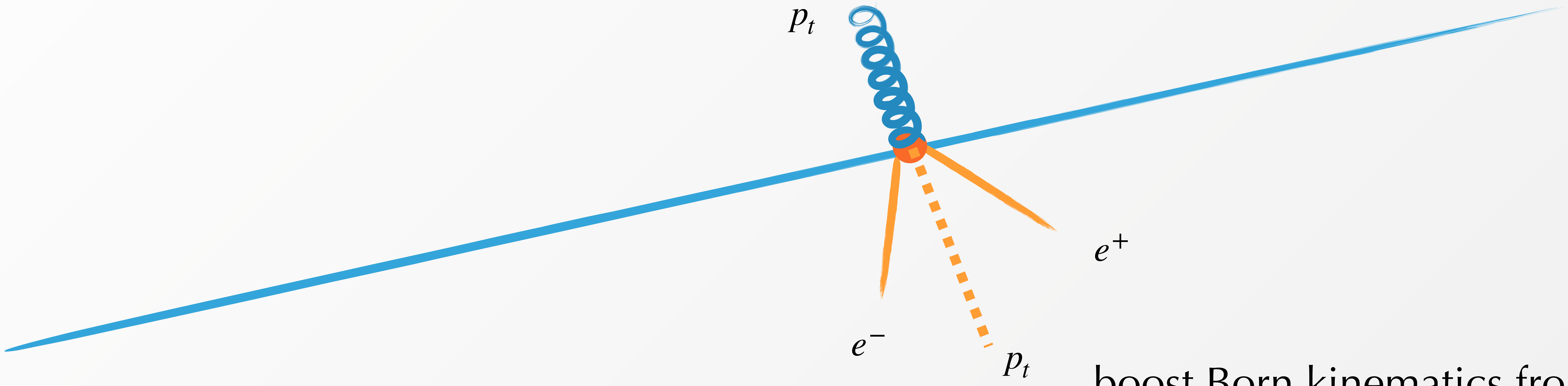
[Catani et al '15]




Generate singlet $p_t$ by QCD radiation

# Inclusion of transverse recoil effects

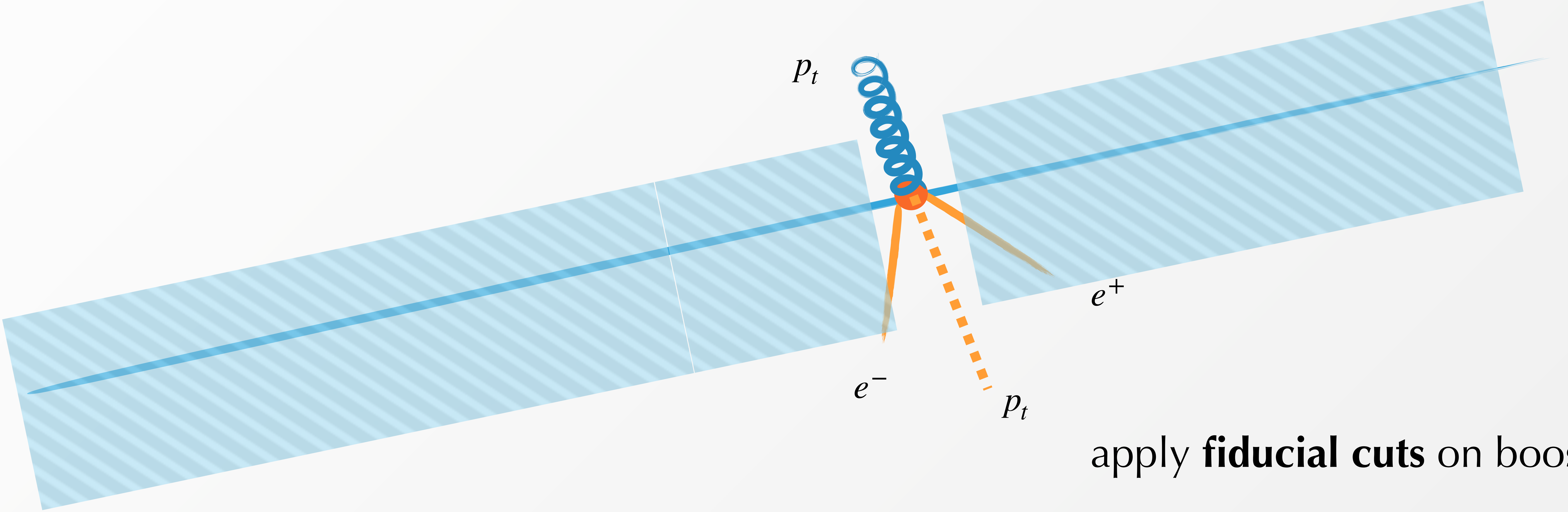
[Catani et al '15]




boost Born kinematics from boson rest frame (e.g. CS) to lab frame with that $p_t$

# Inclusion of transverse recoil effects

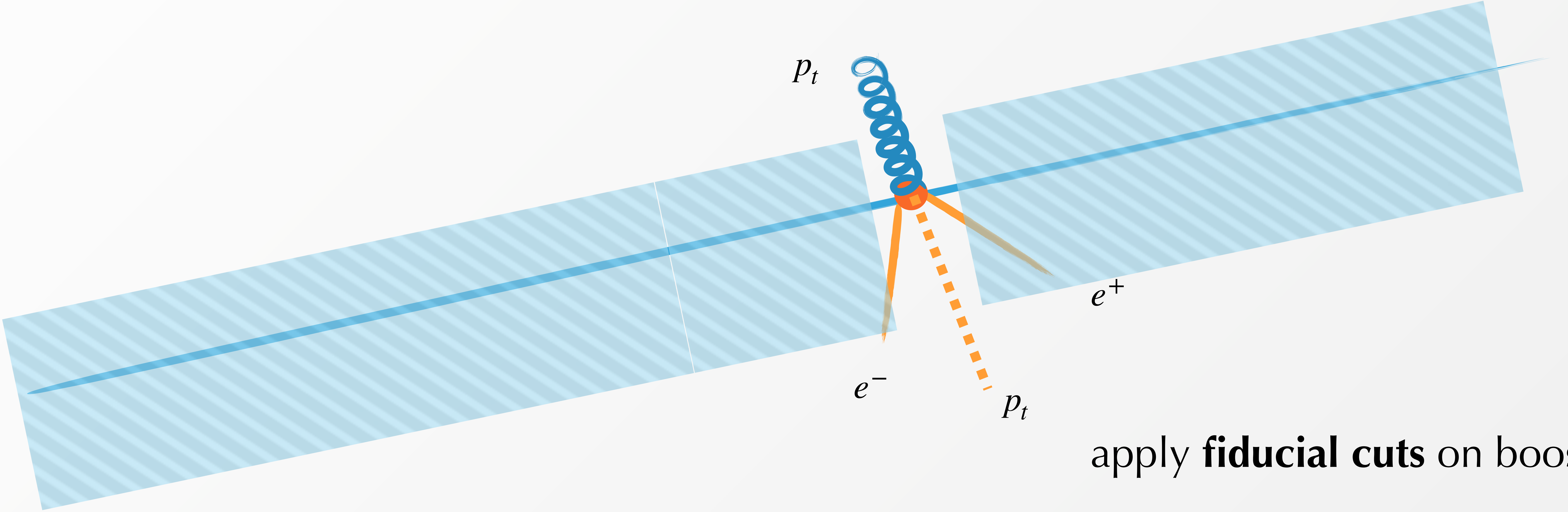
[Catani et al '15]




apply **fiducial cuts** on boosted Born kinematics

# Inclusion of transverse recoil effects

[Catani et al '15]

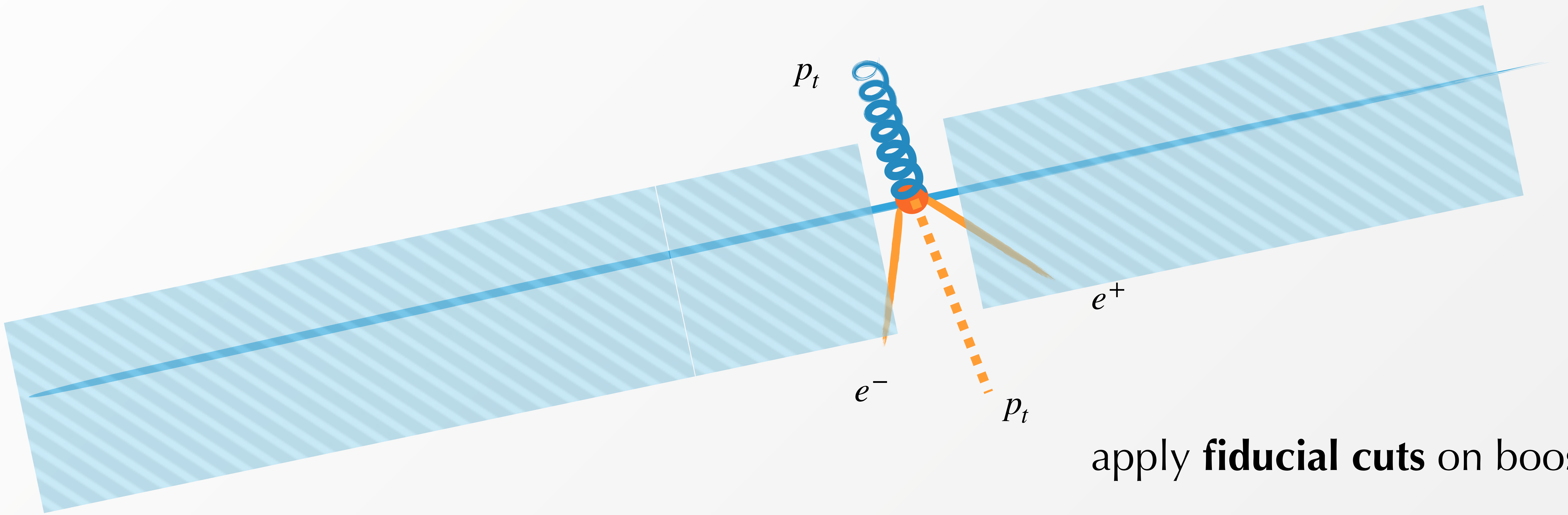



apply **fiducial cuts** on boosted Born kinematics

Sufficient to capture the **full linear fiducial power correction** for $p_t$ [Ebert et al. '20]

# Inclusion of transverse recoil effects

[Catani et al '15]




apply **fiducial cuts** on boosted Born kinematics

Implementation in RadISH:

- Each contribution in the resummation formula boosted in the corresponding frame
- Derivative of the expansion computed on-the-fly, boost computed according to the value of $p_t$

# Results

# Higgs production: setup

Higgs fiducial region defined as [ATLAS 2018]

$$\min(p_t^{\gamma_1}, p_t^{\gamma_2}) > 31.25\ \text{GeV}\,, \qquad \max(p_t^{\gamma_1}, p_t^{\gamma_2}) > 43.75\ \text{GeV}$$

$$0 < |\eta^{\gamma_{1,2}}| < 1.37 \quad \text{or} \quad 1.52 < |\eta^{\gamma_{1,2}}| < 2.37\,, \qquad |Y_{\gamma\gamma}| < 2.37$$

Central scales chosen as

$$\mu_R = \kappa_R\, M_H \qquad \mu_F = \kappa_F\, M_H, \qquad Q = \kappa_Q\, M_H$$

Scale uncertainty:

[canonical 7 scale variation + variation of $\kappa_Q$ by a factor of 2 for central $\mu_R, \mu_F$] $\times$ 3 matching scale choices $\rightarrow$ **27 variations**

PDF4LHC15 NNLO parton densities. NNLO predictions from NNLOJET

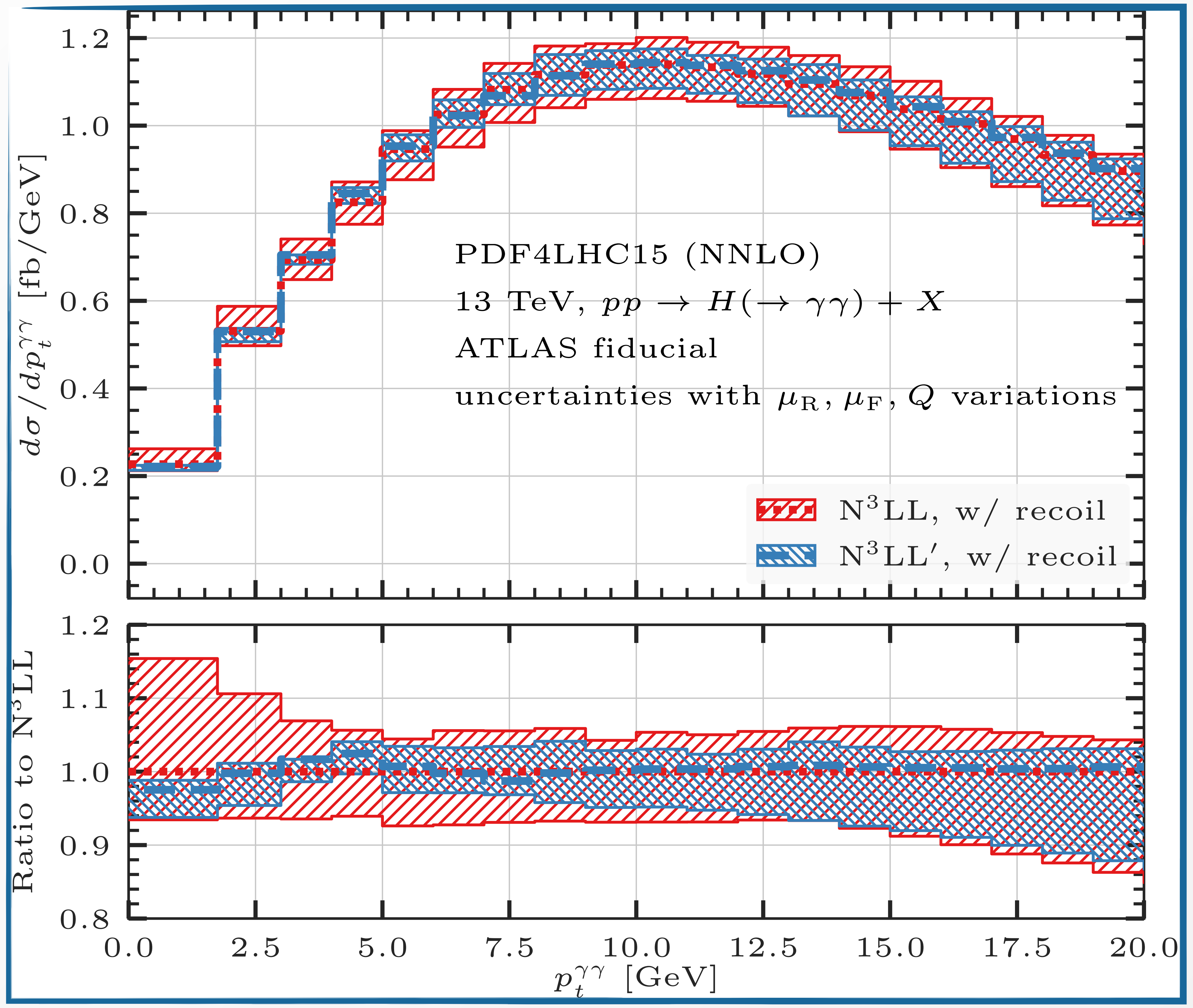
# Higgs production: N³LL′ effects

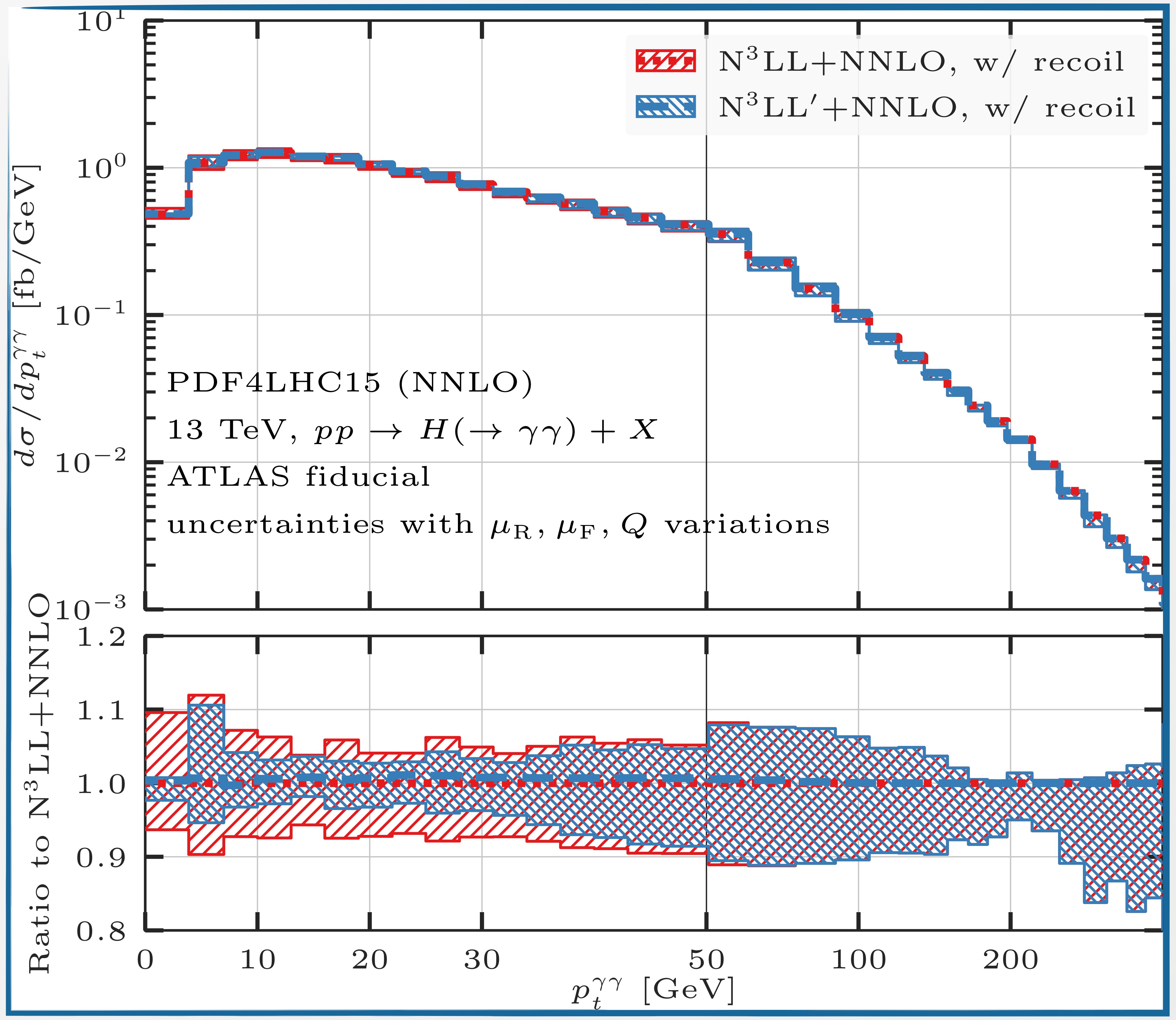
Significant reduction in theoretical uncertainty below 15 GeV, especially **below 5 GeV**

**Central value almost unchanged** between N³LL and N³LL′

Reduction in scale uncertainty limited at matched level (**statistical fluctuations** of the fixed order at small $p_t$)

$\kappa_R = \kappa_F = \kappa_Q = 1/2$

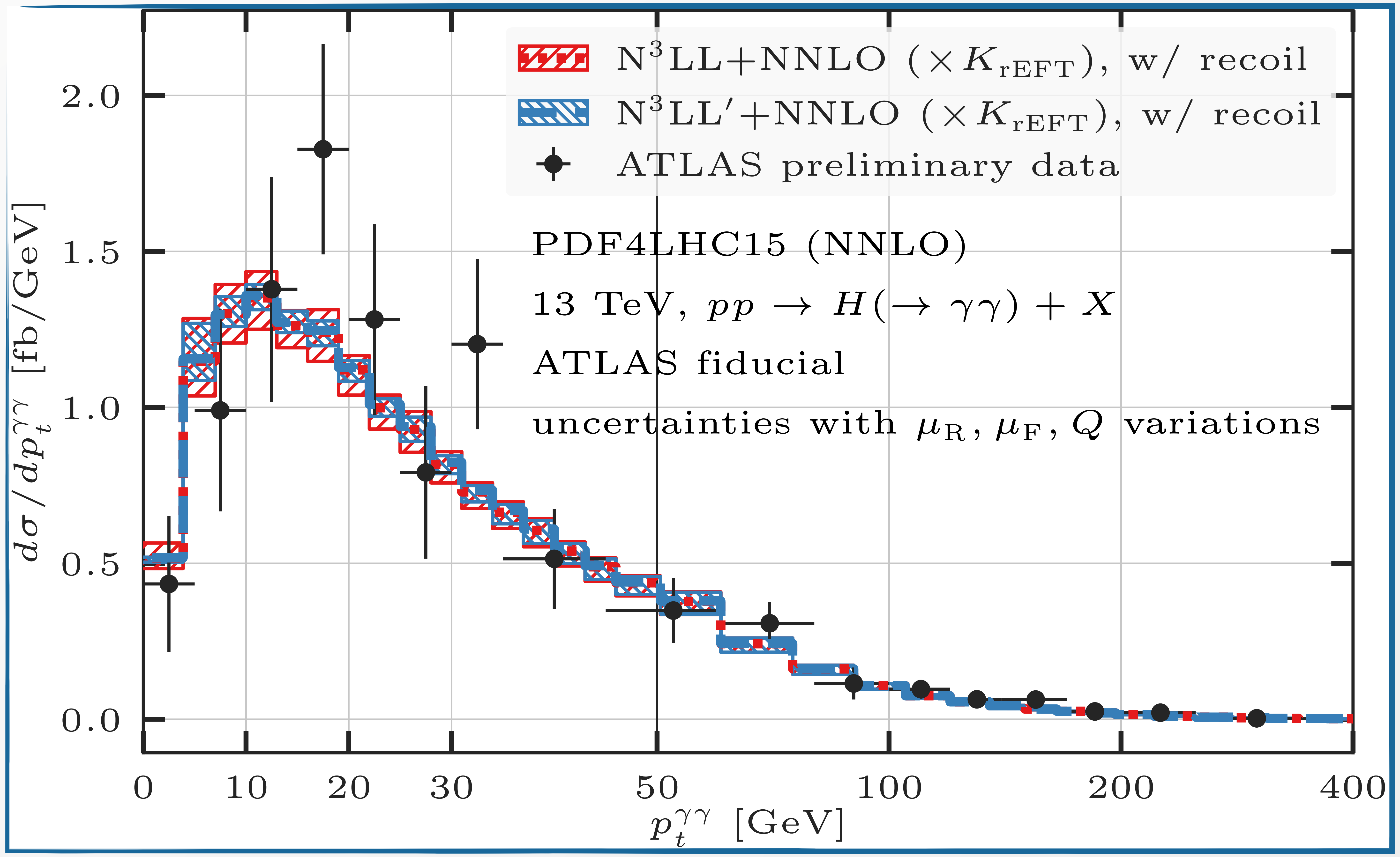


# Higgs production: comparison with ATLAS data

ATLAS preliminary data from https://cds.cern.ch/record/2682800

Theoretical predictions rescaled by $K_{\text{rEFT}} = 1.06584$ to account for exact LO top-mass dependence



# Recapitulation and outlook

- Results for Higgs $p_t$ at N$^3$LL′+NNLO accuracy by including **all constant terms of relative order** $\alpha_s^3$ in the RadISH formalism
- RadISH now includes **recoil effects** which improve the description of decay kinematics in the fiducial region
- Precise theoretical prediction **in the fiducial region** for $H \to \gamma\gamma$
- Reduction of theoretical uncertainty at N$^3$LL′. Resummation uncertainty at the 5-10% level
- Marginal effect of recoil in matched results

# Backup

# Matching to fixed order

Two different families of **matching schemes**, defined at the **differential** level (due to the inclusion of **recoil effects**)

**Additive matching**

$$\frac{d\Sigma_{\rm add}^{{\rm N}^k{\rm LL}^{(\prime)}}(v)}{dv} = \left(\frac{d\Sigma^{{\rm N}^k{\rm LL}^{(\prime)}}(v)}{dv} - \frac{d\Sigma_{\rm exp}^{{\rm N}^k{\rm LL}^{(\prime)}}(v)}{dv}\right) Z(v) + \frac{d\Sigma^{{\rm N}^{k-1}{\rm LO}}(v)}{dv}$$

**Multiplicative matching**

$$\frac{d\Sigma_{\rm mult}^{{\rm N}^k{\rm LL}^{(\prime)}}(v)}{dv} = \left(\frac{d\Sigma^{{\rm N}^k{\rm LL}^{(\prime)}}(v)/dv}{d\Sigma_{\rm exp}^{{\rm N}^k{\rm LL}^{(\prime)}}(v)/dv}\right)^{Z(v)} \frac{d\Sigma^{{\rm N}^{k-1}{\rm LO}}(v)}{dv}$$

At NNLO+N$^3$LL$'$ the two matching schemes are on equal footing, differences starts at $\alpha_s^4$

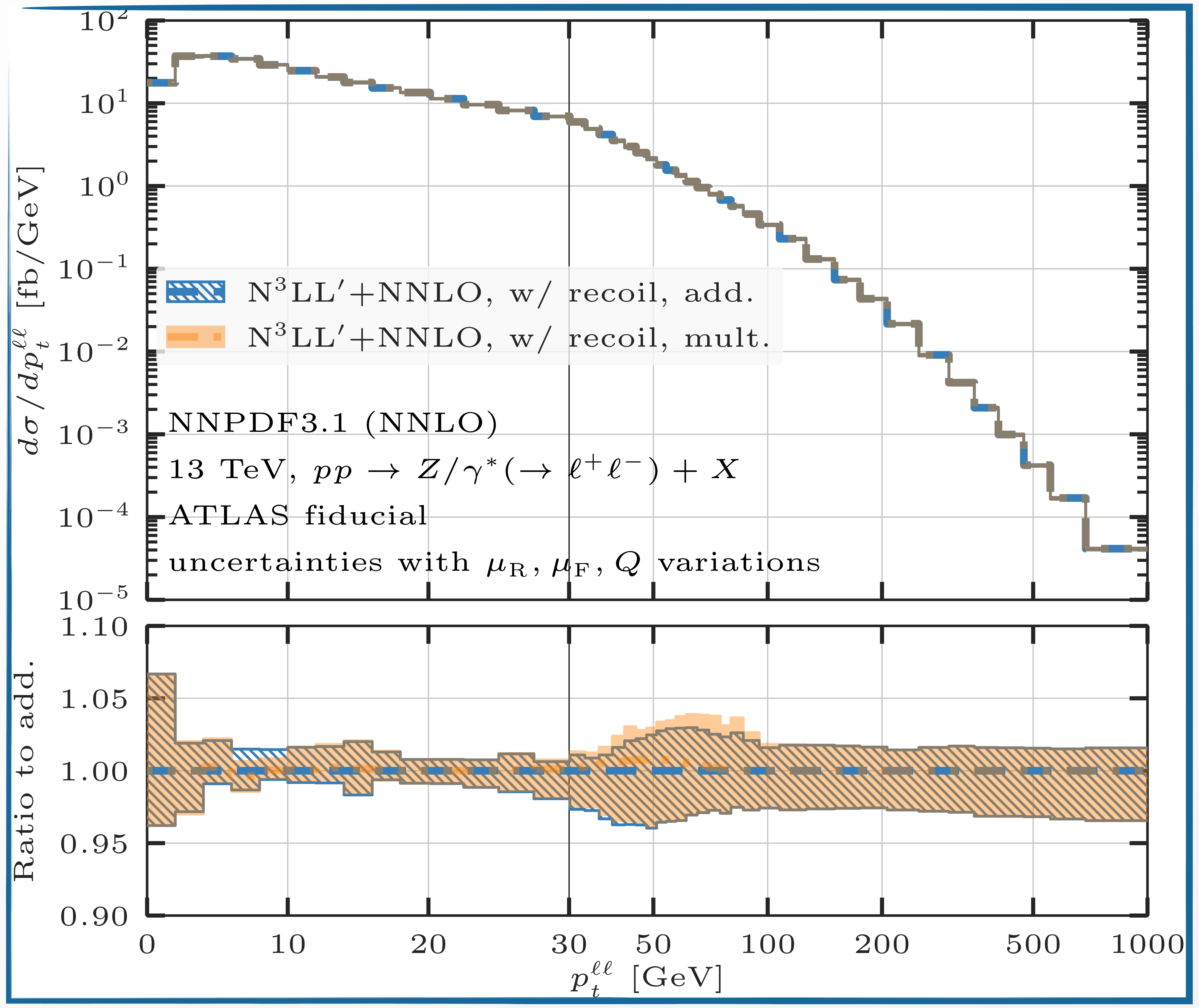
Damping function (does not act on linear power corrections)

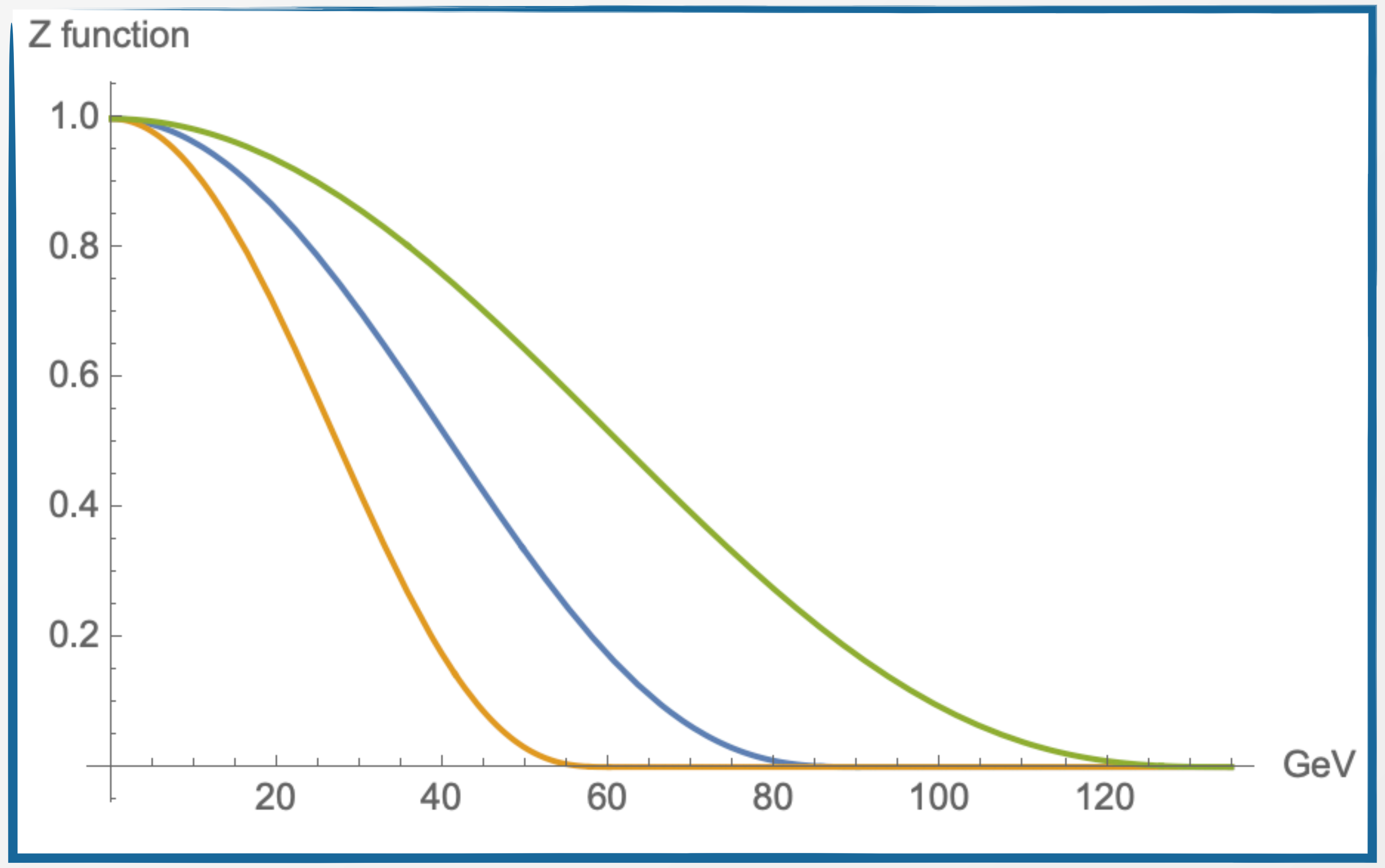
$$Z(v) = \left[1 - (v/v_0)^2\right]^3 \Theta(v_0 - v)$$

$v_0$ varied in the interval [2/3, 3/2] around central value to **estimate matching uncertainty**

Central value $v_0 = 1$ for $p_\perp$

# Matching systematics

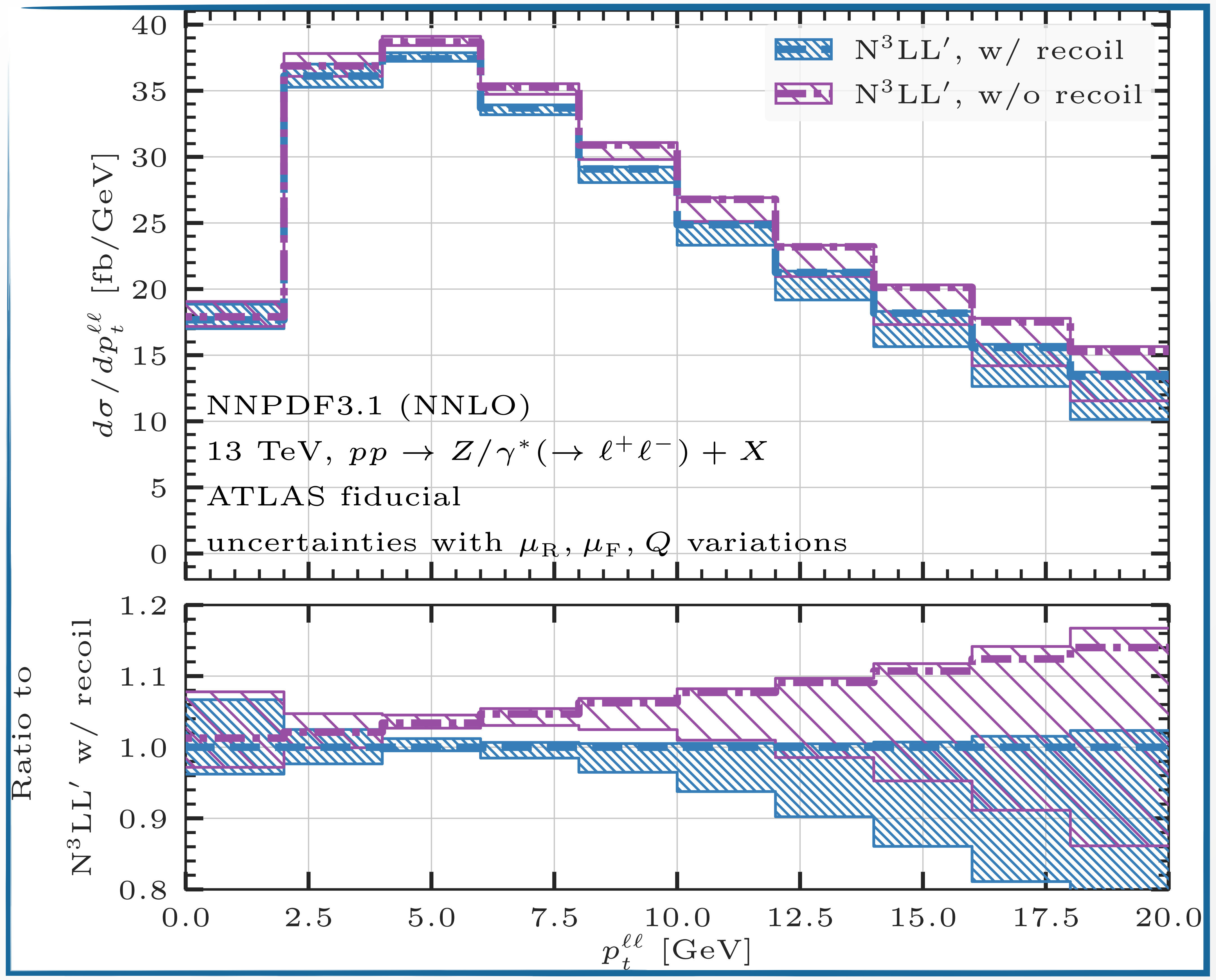


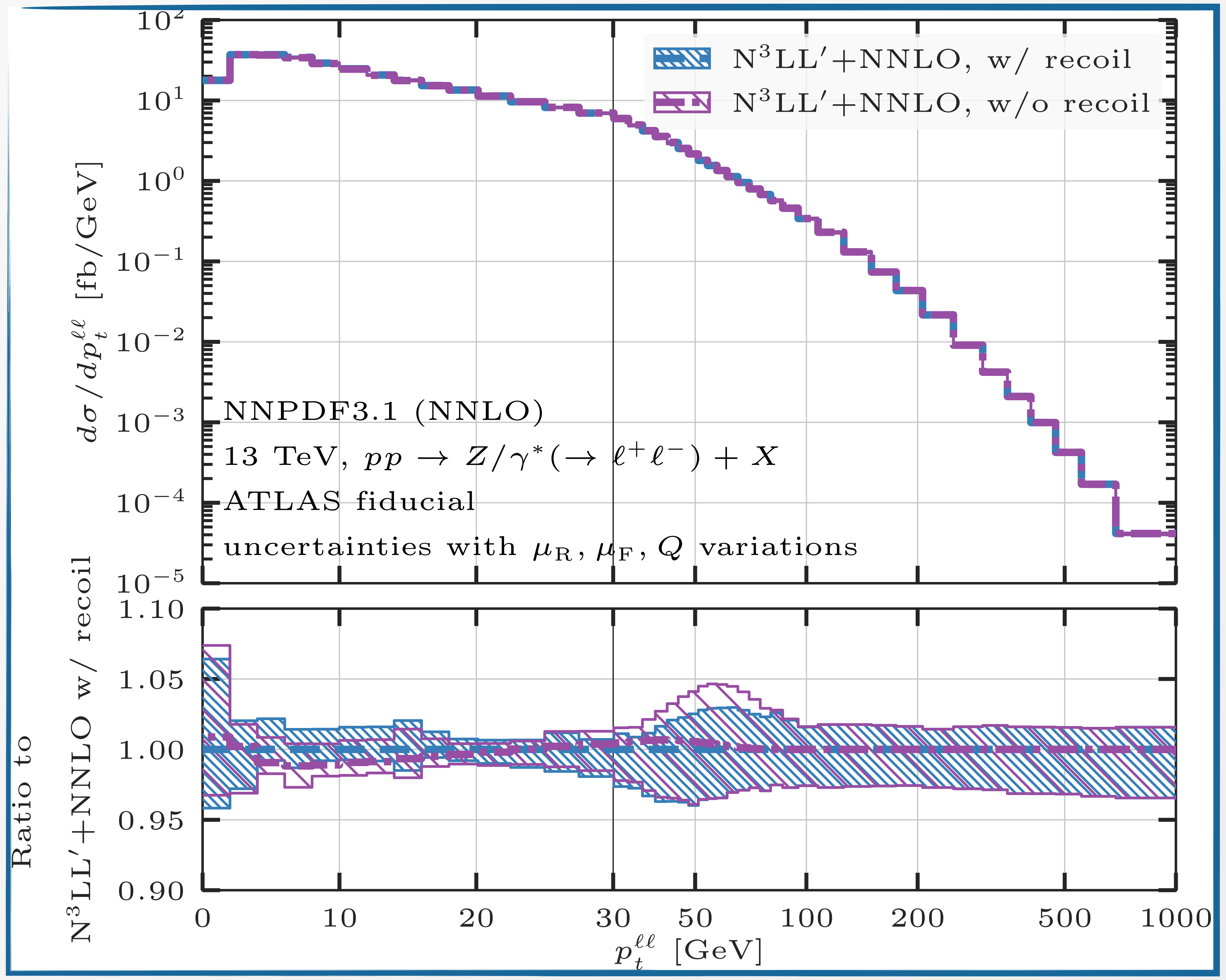




Very mild matching scheme dependence both for central results and uncertainties

Additive matching uncertainty band reliably estimate matching ambiguities

# Transverse recoil effects in fiducial DY setup




At the pure resummed level recoil prescription captures whole linear power corrections from fiducial cuts

Effect reduce at 1-2% level after matching to fixed order (effect becomes $\mathcal{O}(\alpha_s^4)$)

$$\ln(Q/k_{t1}) \to 1/p\,\ln(1+(Q/k_{t1})^p)$$

Pure resummed: band widening due to power corrections due to modified logs

$$\int_0^M \frac{dk_{t1}}{k_{t1}} \to \int_0^\infty \frac{dk_{t1}}{k_{t1}} \frac{(Q/k_{t1})^p}{1+(Q/k_{t1})^p}$$

# Ambiguity in the definition of primed accuracy

$$\mathscr{L}_{\rm NNLL}(k_{t1}) = \sum_{c,c'} \frac{d|\mathscr{M}_B|^2_{cc'}}{d\Phi_B} \sum_{i,j} \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} f_i\left(k_{t1}, \frac{x_1}{z_1}\right) f_j\left(k_{t1}, \frac{x_2}{z_2}\right)$$

$$\times \left\{ \delta_{ci}\,\delta_{c'j}\,\delta(1-z_1)\,\delta(1-z_2)\left(1 + \frac{\alpha_s(\mu_R)}{2\pi} H^{(1)}(\mu_R)\right) + \frac{\alpha_s(\mu_R)/(2\pi)}{1 - 2\alpha_s(\mu_R)\beta_0 \ln(\mu_R/k_{t1})}\left(C^{(1)}_{ci}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1, c, i \leftrightarrow z_2, c', j\}\right) \right\}$$

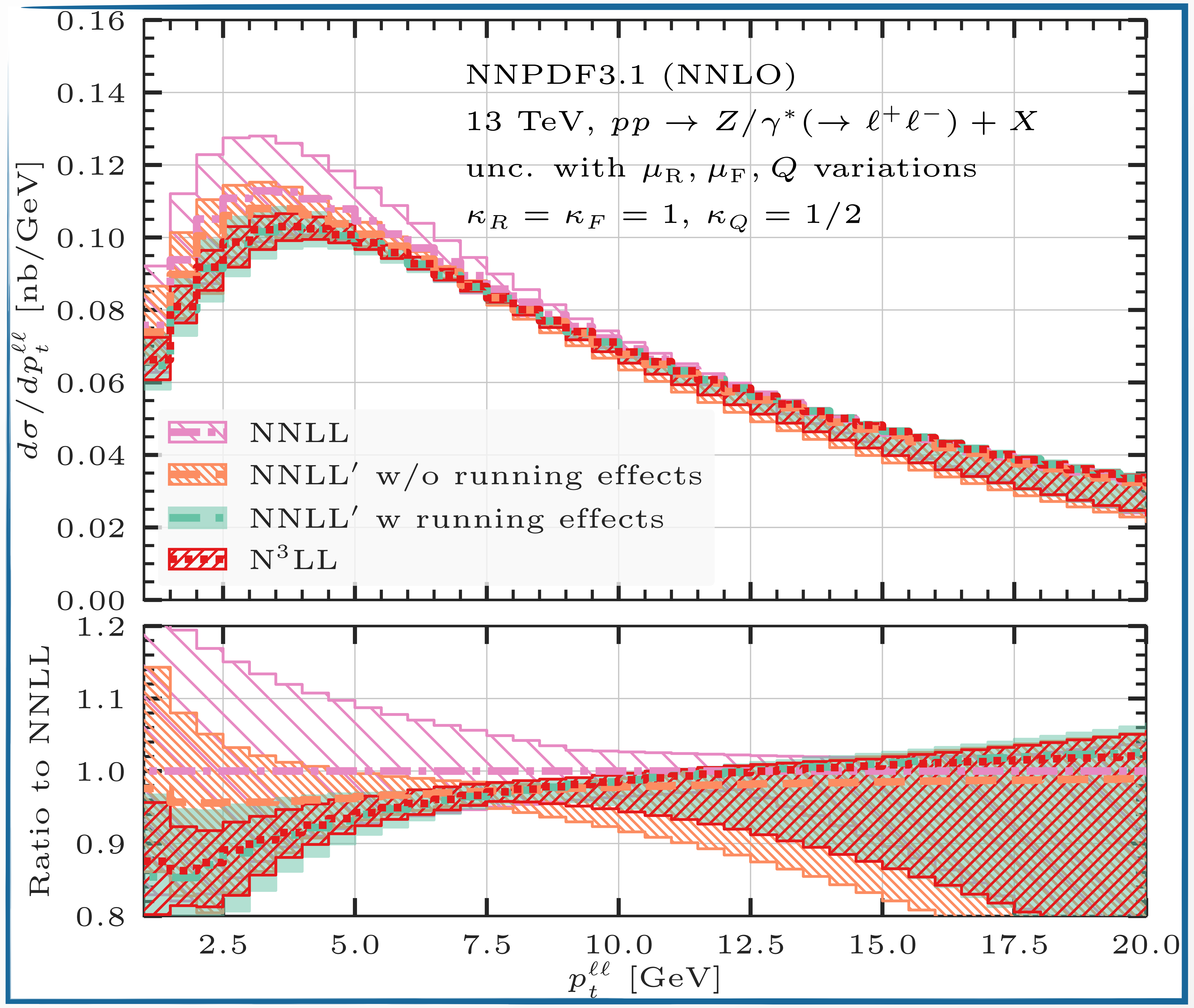
Scale at which the $\alpha_s^k$ term is evaluated is subleading at N$^k$LL′ accuracy

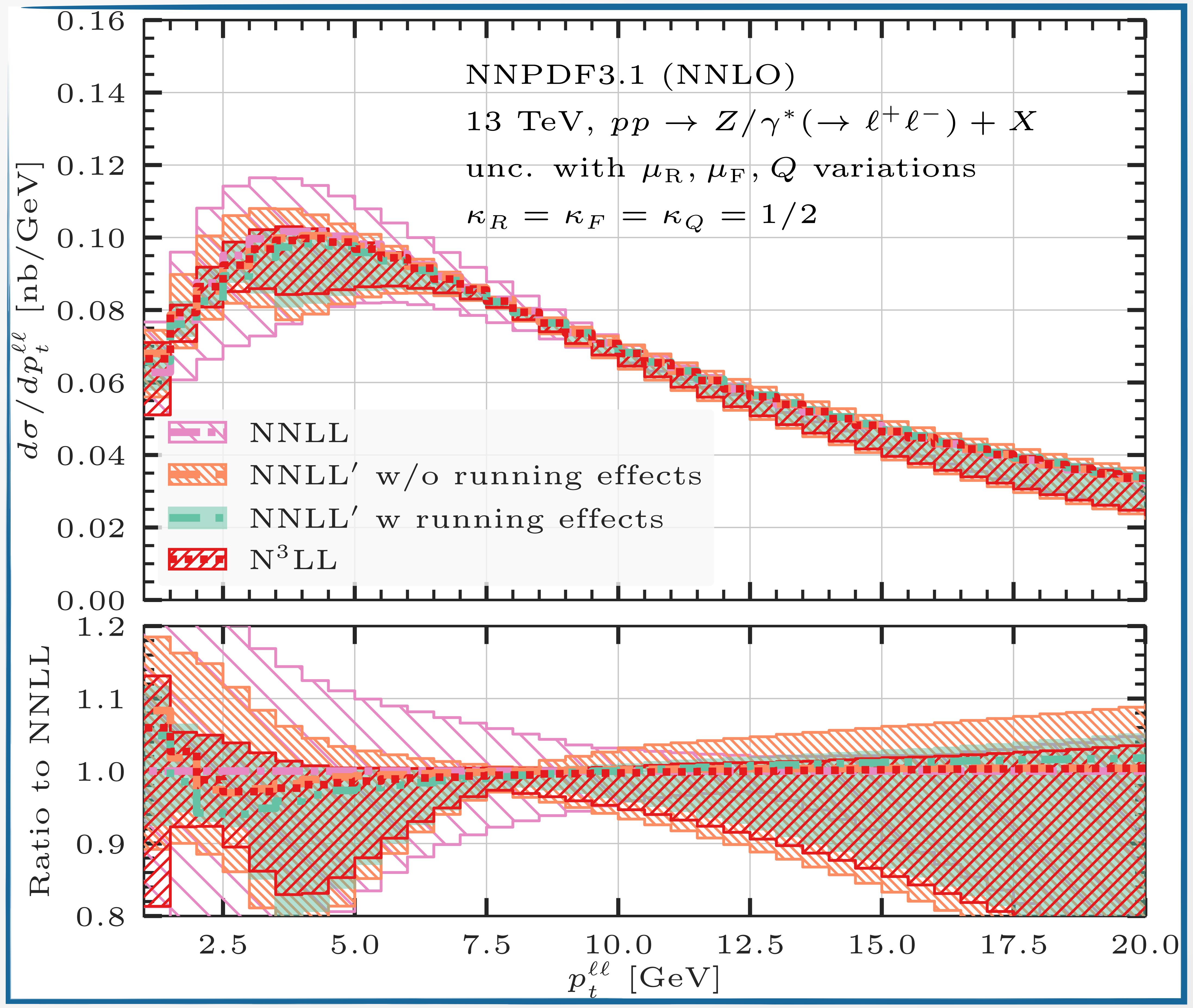
One can evaluate this contribution with $\alpha_s(\mu_R)$ rather → difference reflects **ambiguity** of these subleading effects

NLL′ **with running**: $\mathscr{L}_{\rm NLL'} = \mathscr{L}_{\rm NNLL}$ **Our default choice**

NLL′ **without running**: $\mathscr{L}_{\rm NLL'} = \mathscr{L}_{\rm NNLL}$ with $\alpha_s(\mu_R)$ in the $C_1$ component

# Ambiguity in the definition of primed accuracy




NNLL′ with and without running closer to $N^3LL$ than NNLL is

NNLL′ with running band in better agreement to $N^3LL$: $N^3LL$ contained within NNLL′ with running uncertainty

Band for NNLL′ with running covers difference between two NNLL′ → reliable estimate of prime ambiguity