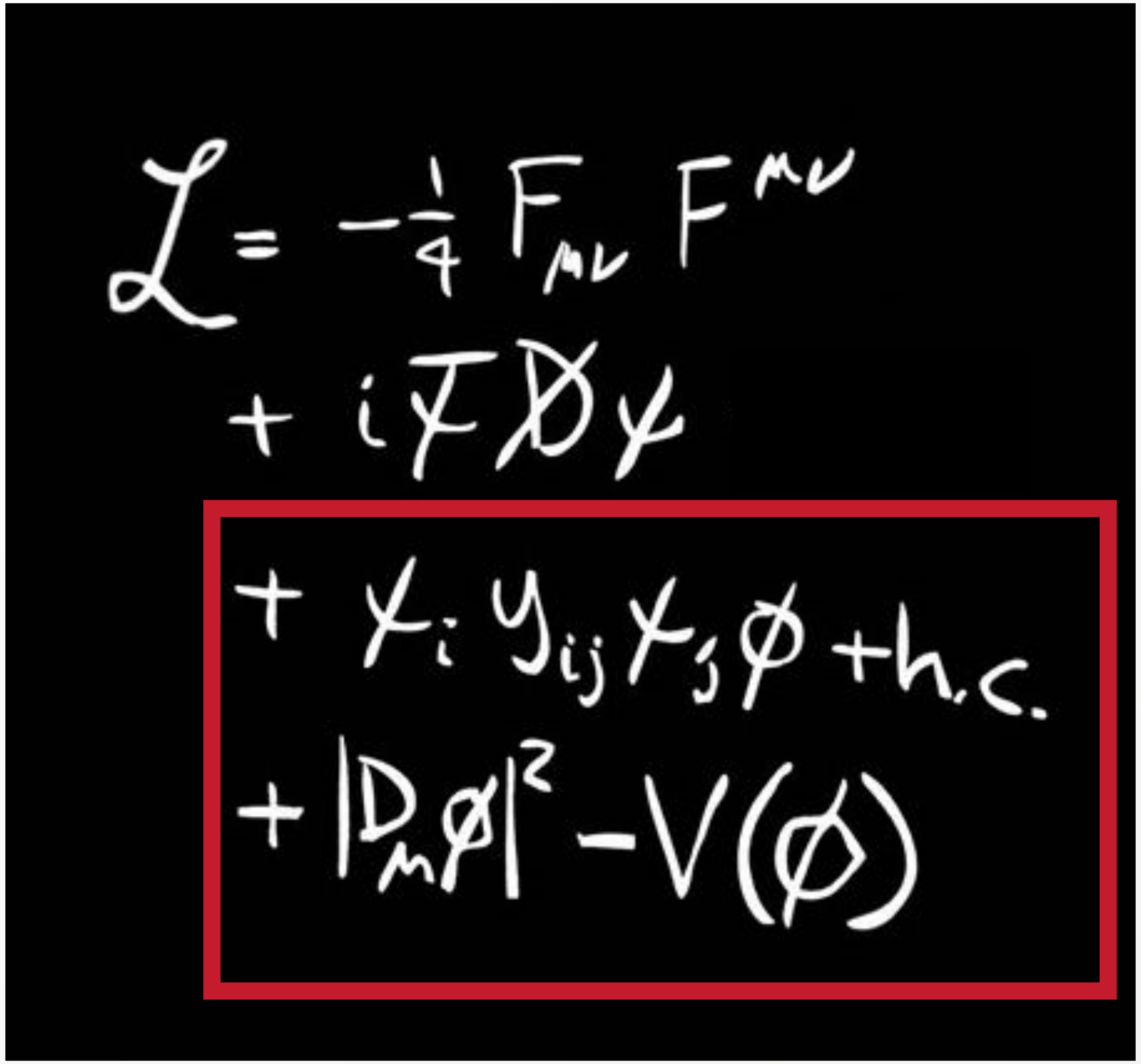
Higgs transverse momentum with a jet veto: a double-differential resummation

Luca Rottoli

University of Milan-Bicocca & LBNL

Based on 1909.04704 with P. Monni and P. Torrielli

The Higgs boson sector



Various questions still unanswered

- What is the form of the Higgs potential?
- Is it really ϕ^4?
- Are Yukawa couplings responsible for all fermion masses (5 orders of magnitude)?
- …

Higgs sector needs stress-testing

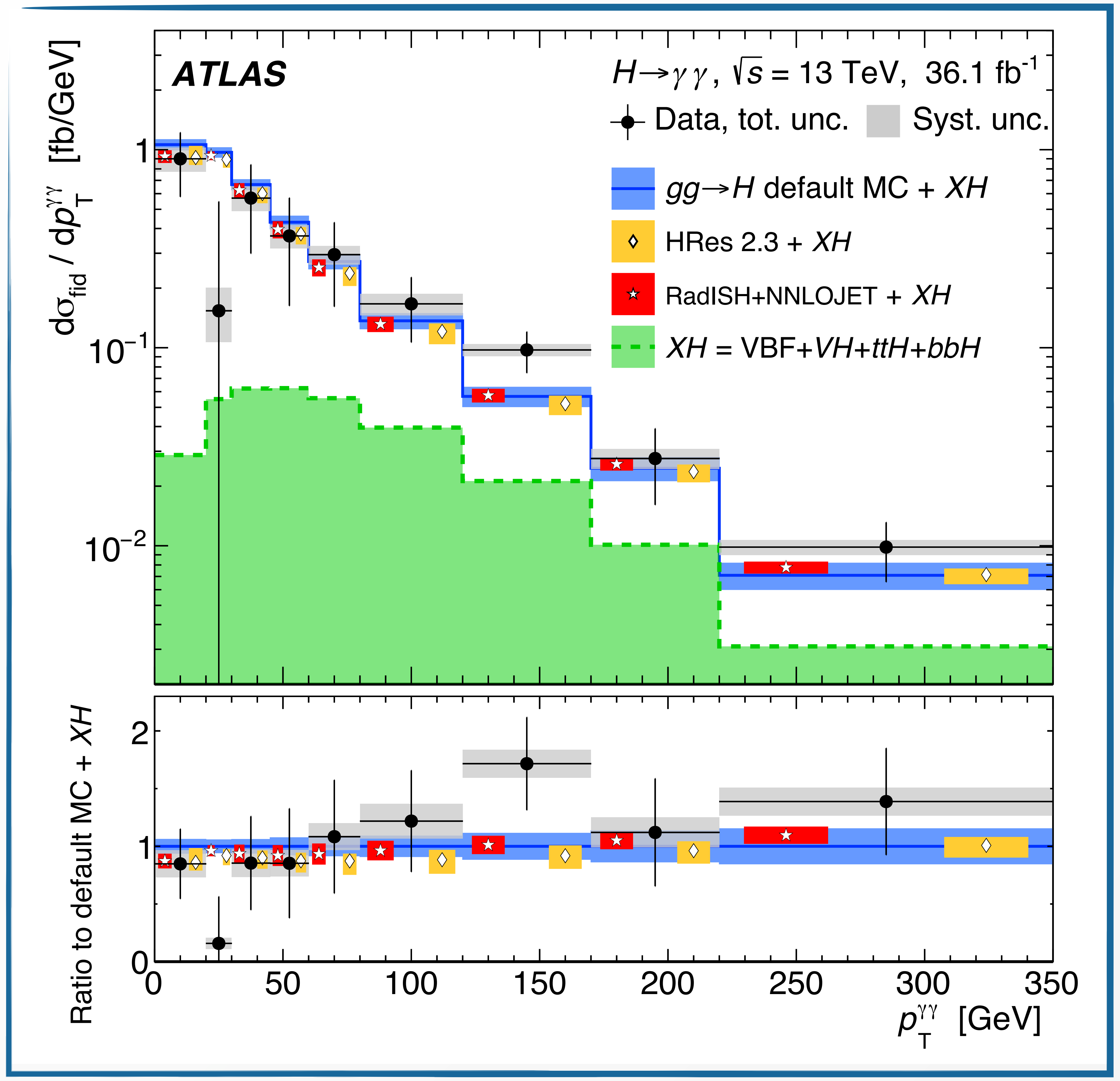
How can we stress-test the Higgs sector?

$$g = g_{\text{SM}}(1 + \delta)$$

$$\delta \sim \mathcal{O}\left(\frac{v^2}{\Lambda_{\text{NP}}^2}\right)$$

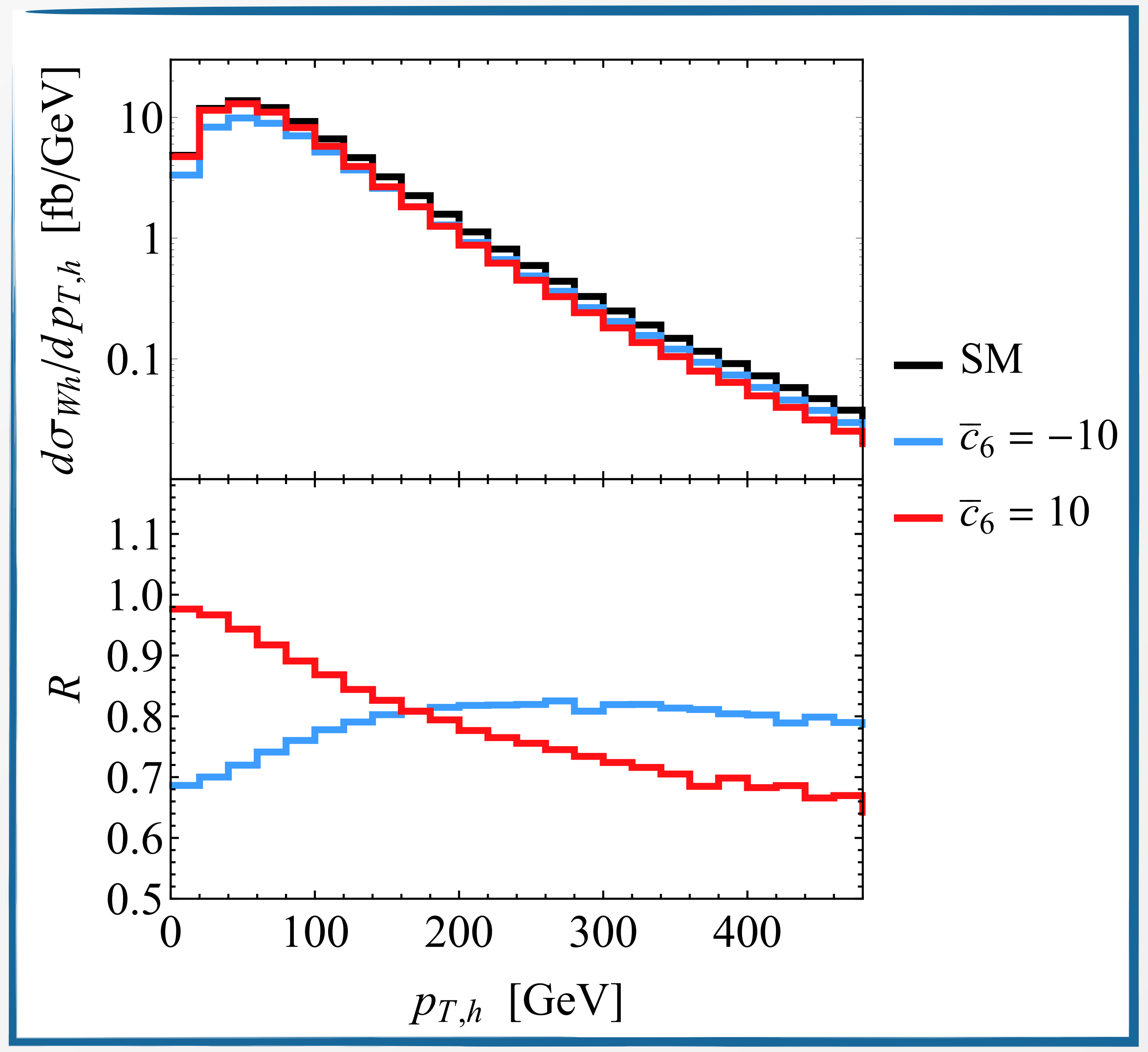
Higher precision → sensitivity to small deviations → higher scales being probed

The Higgs transverse momentum

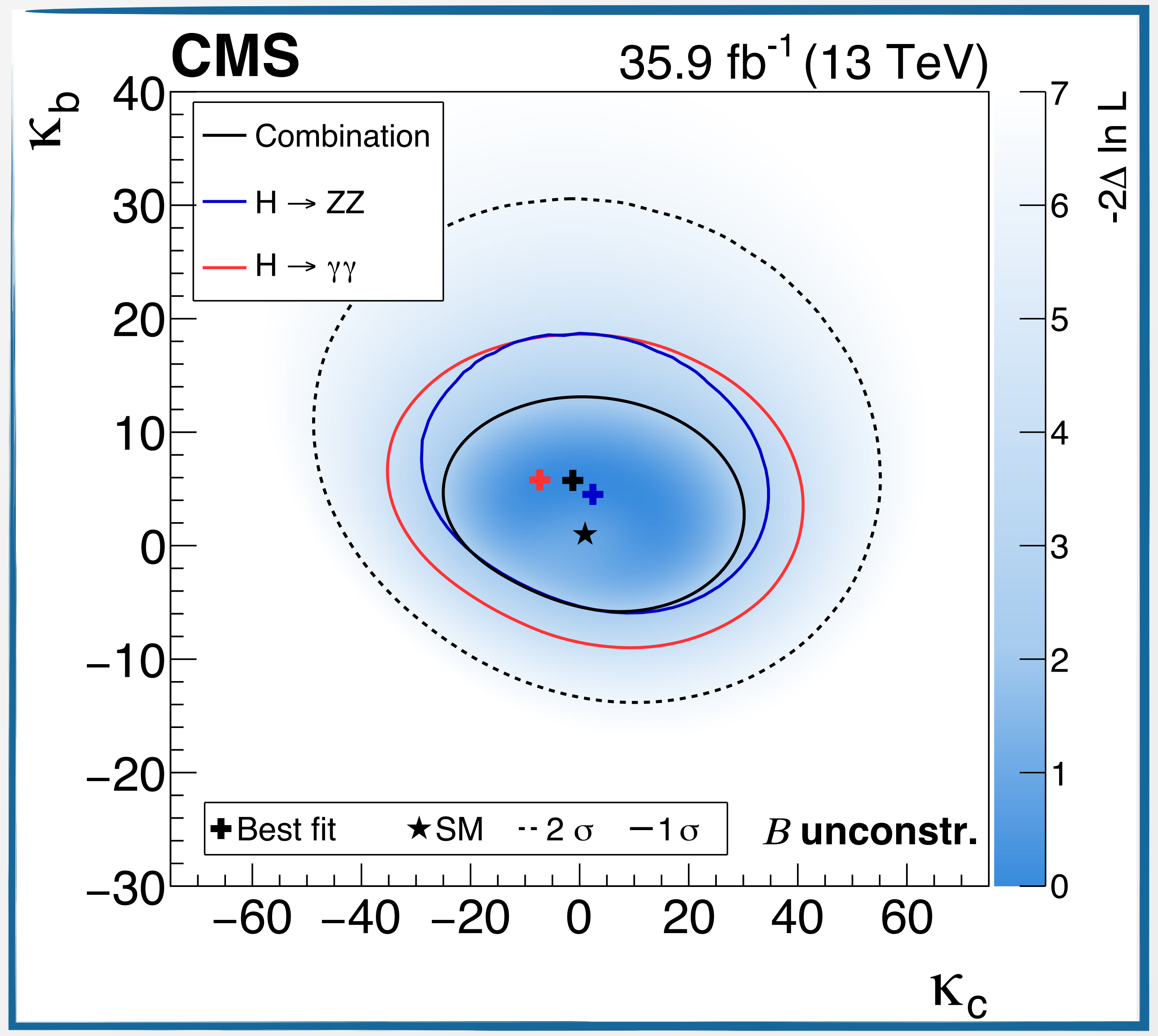


[ATLAS 1802.04146]

- Relatively easy to measure
- Sensitivity to New Physics (e.g. **light Yukawa** couplings, **trilinear** Higgs self-coupling)

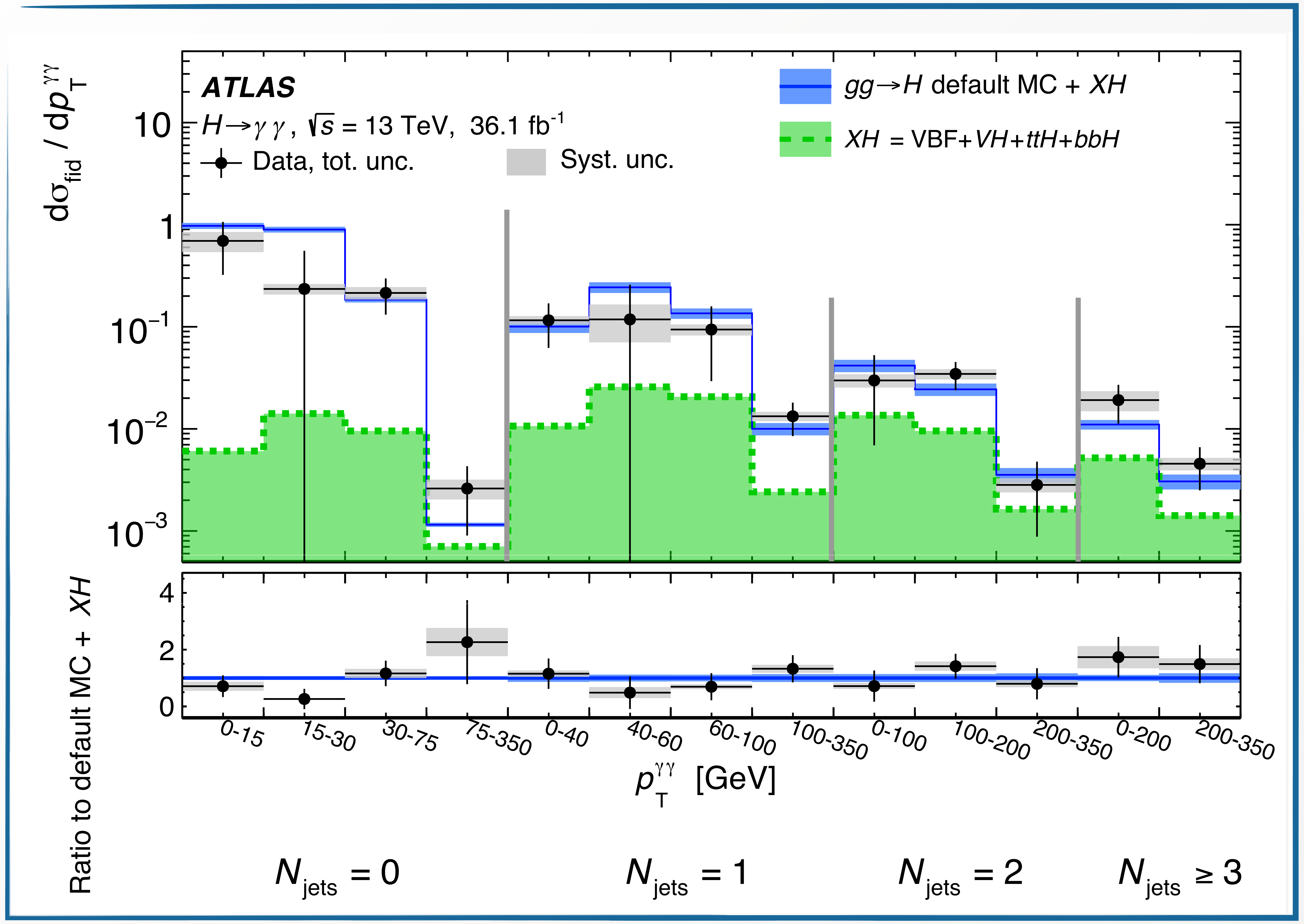


[Bizon *et al.* 1610.05771]



[CMS 1812.06504]
[Bishara *et al.* '16][Soreq *et al.* '16]

The Higgs transverse momentum



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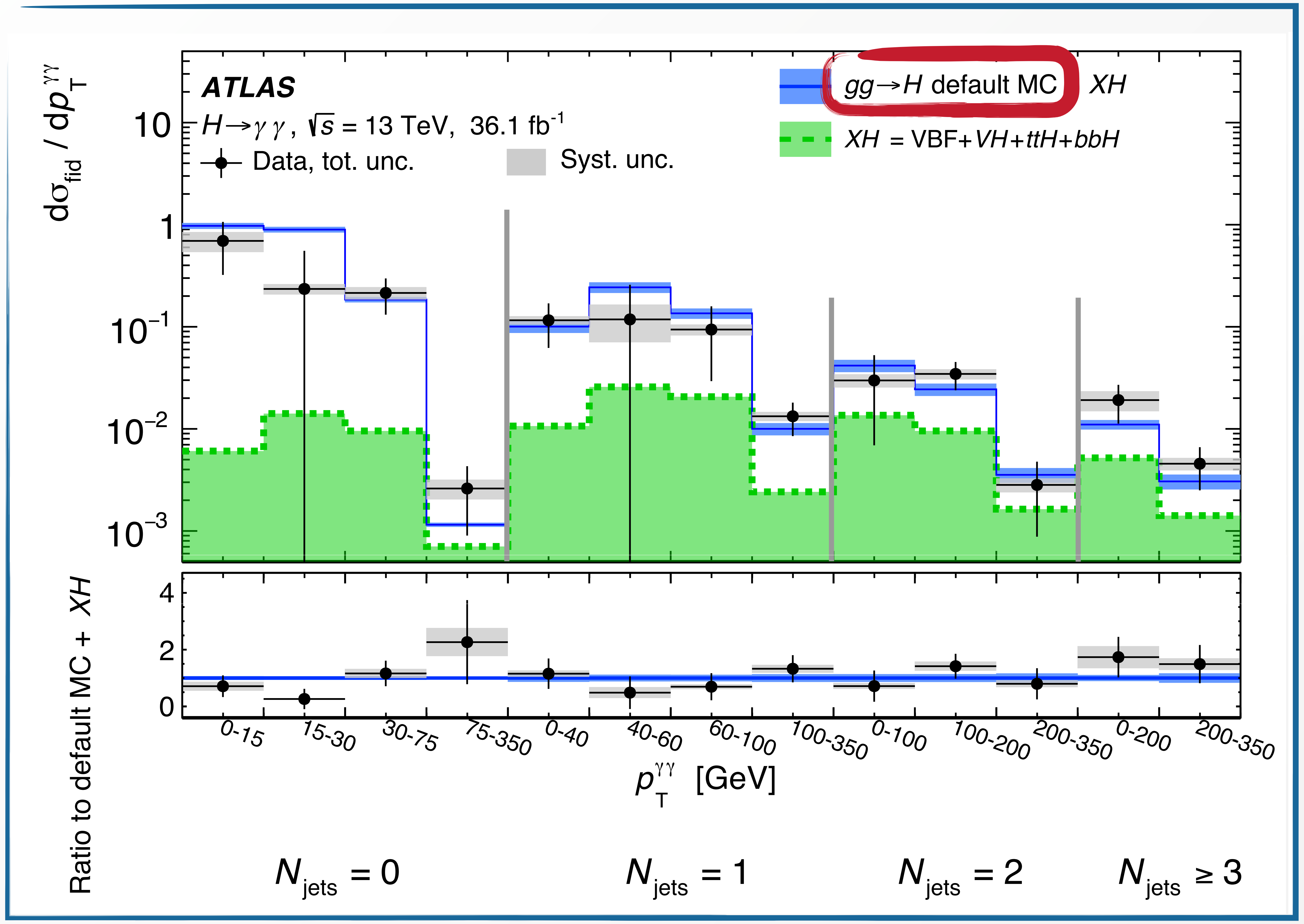
- Experimental analyses categorize events into **jet bins** according to the jet multiplicity
- Increased sensitivity to Higgs boson kinematics, spin-CP properties, BSM effects…
- Similar comments apply also to other analyses (e.g. VH with boosted Higgs, W^+W^- production…)

The Higgs transverse momentum



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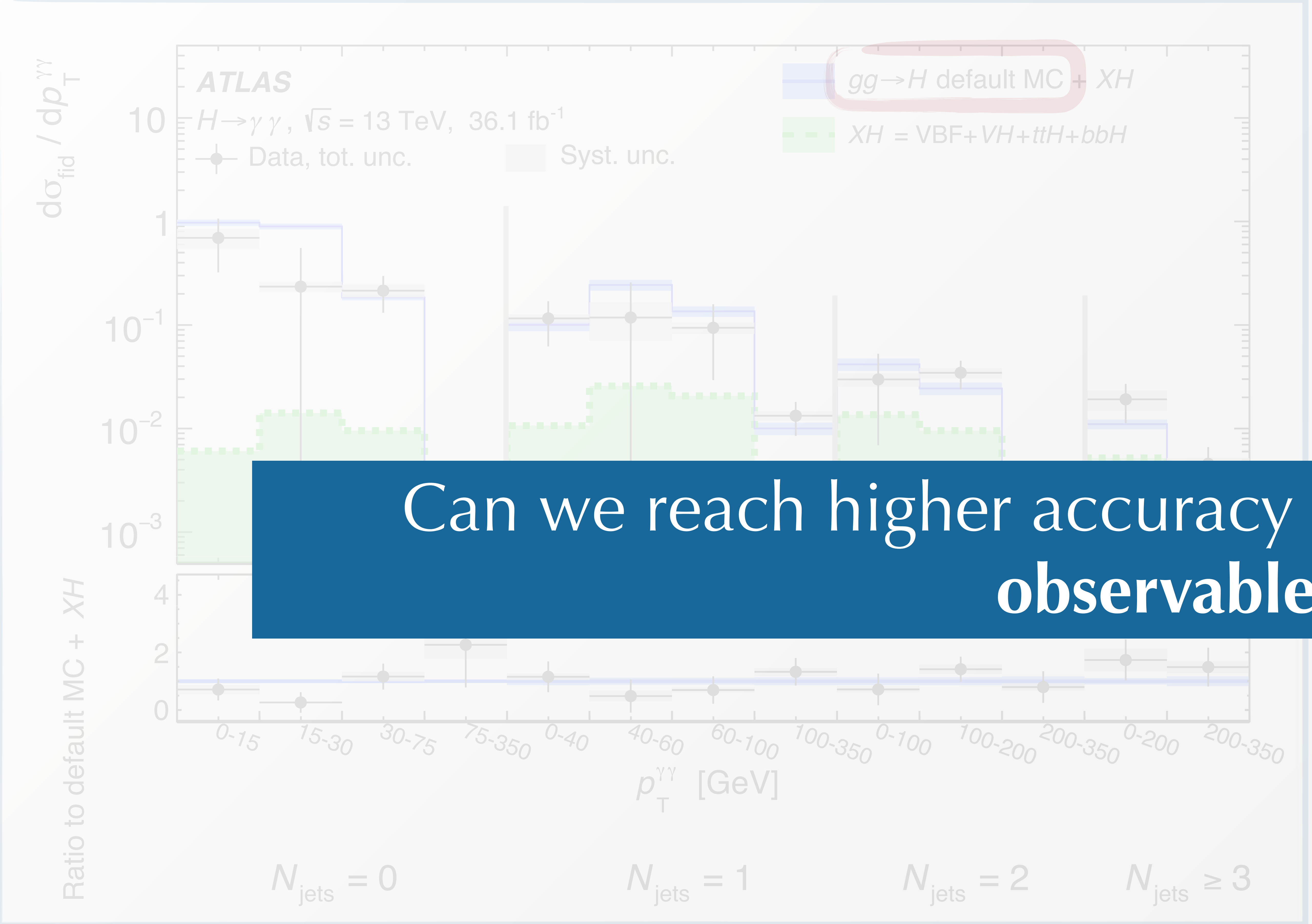

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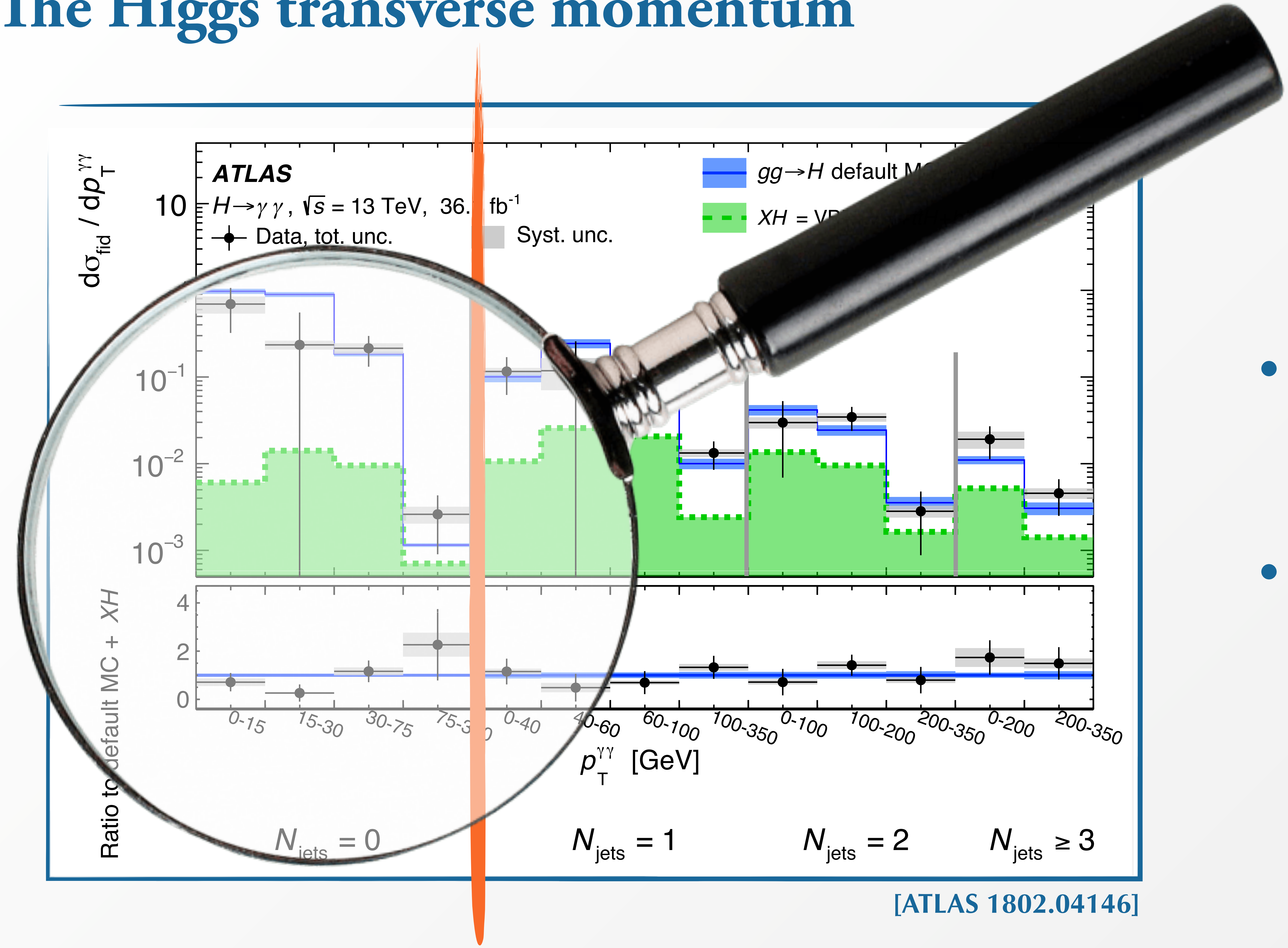
Can we reach higher accuracy for **double-differential observables?**

The Higgs transverse momentum

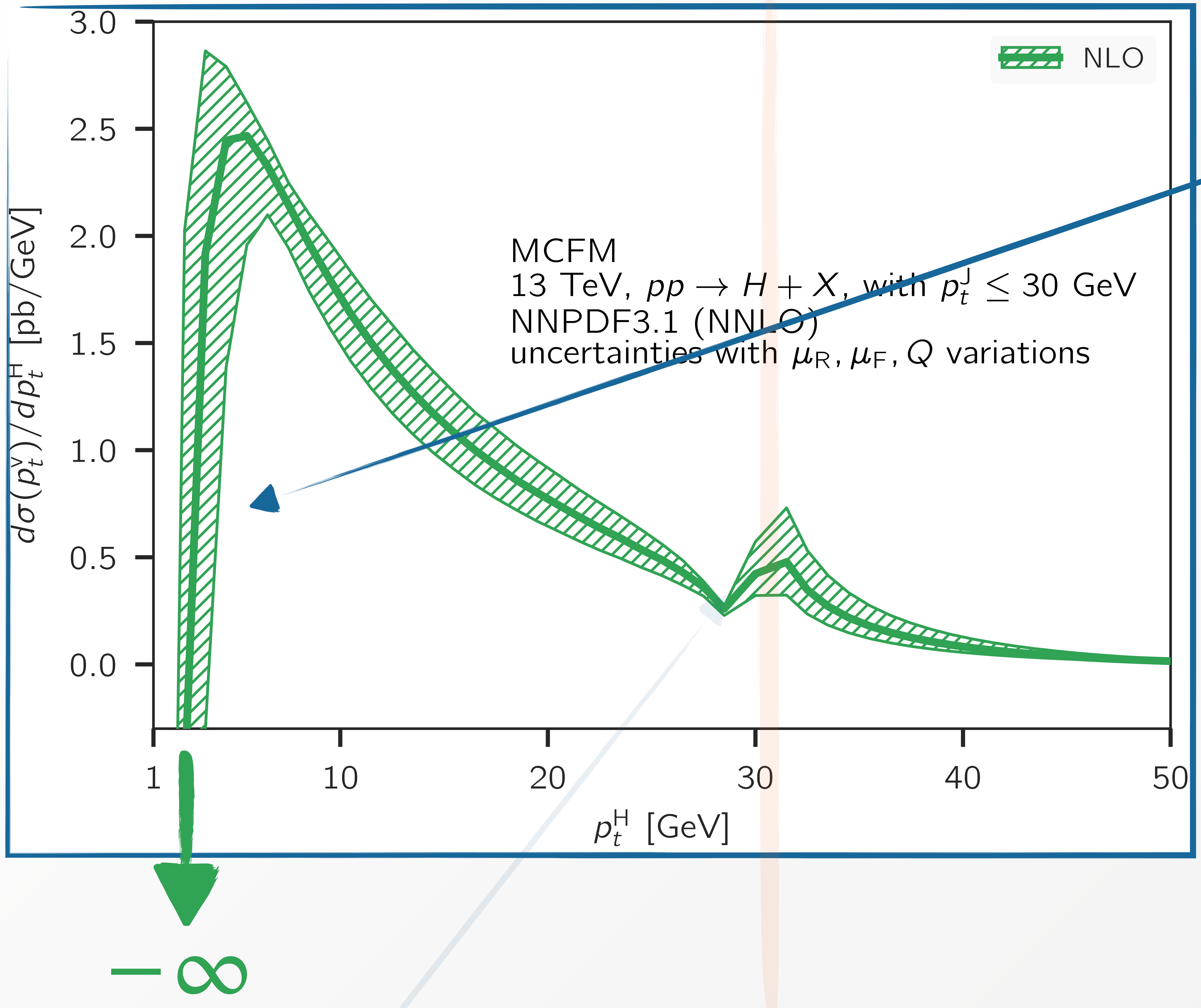


[ATLAS 1802.04146]

$p_\perp^J \leq 30\,\mathrm{GeV}$

- Focus on the **zero-jet bin** $p_\perp^J \leq p_\perp^{\mathrm{J,v}}$
- Jet veto enforced to enhance the Higgs signal with respect to its backgrounds (e.g. W^+W^- event selection) or study of different production channels (e.g. STXS)

The appearance of large logarithms



Large **transverse momentum** logarithms

$$L = \ln(p_\perp^H / m_H) \qquad p_\perp^H \ll m_H$$

Large(ish) **jet veto** logarithms

$$L = \ln(p_\perp^{J,v} / m_H) \qquad p_\perp^{J,v} < m_H$$

Jet veto = 30 GeV

$L = \ln(|\vec{p}_\perp^{\,H} + \vec{p}_\perp^{\,J}| / p_\perp^H)$ Sudakov shoulder logarithms

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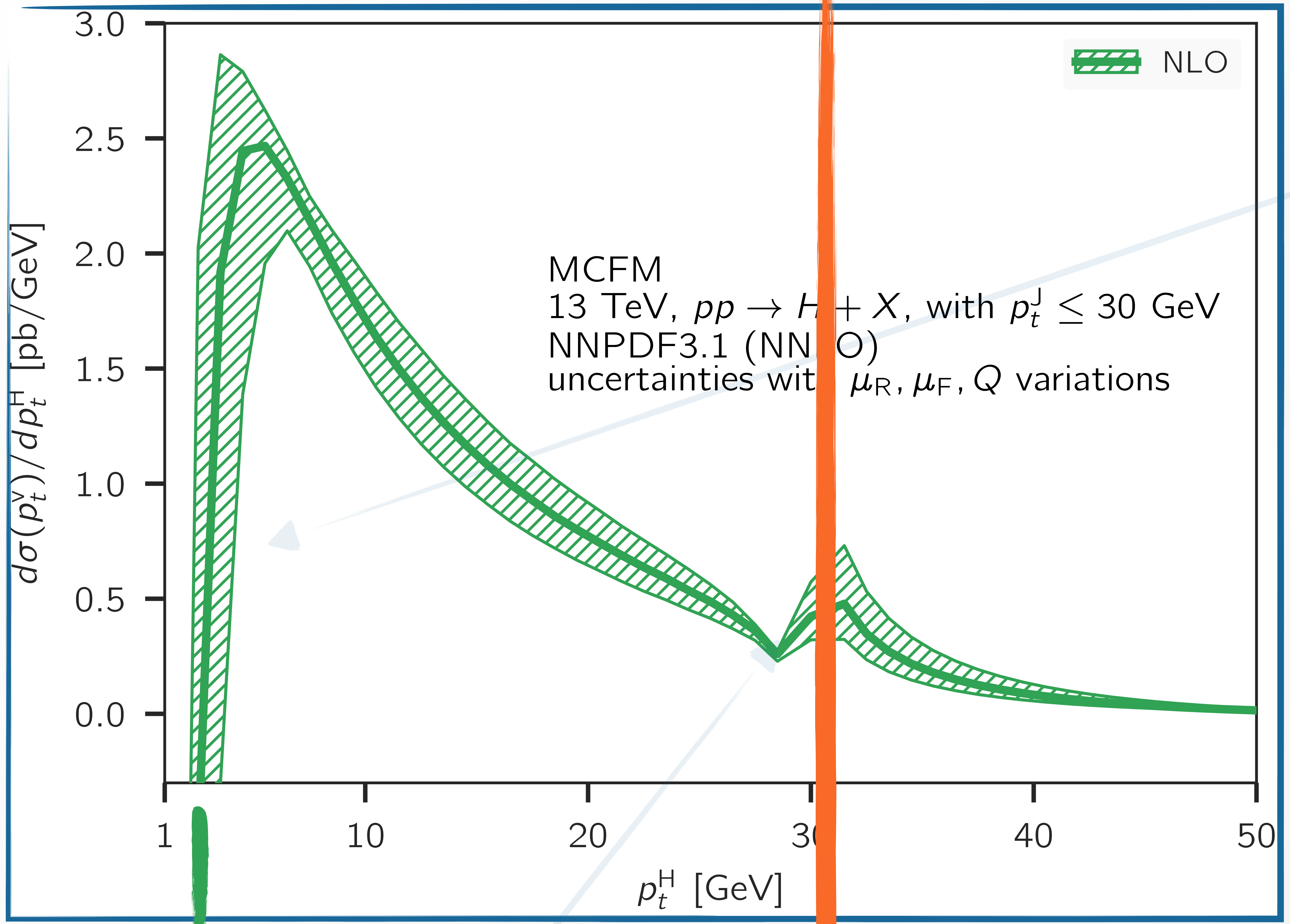

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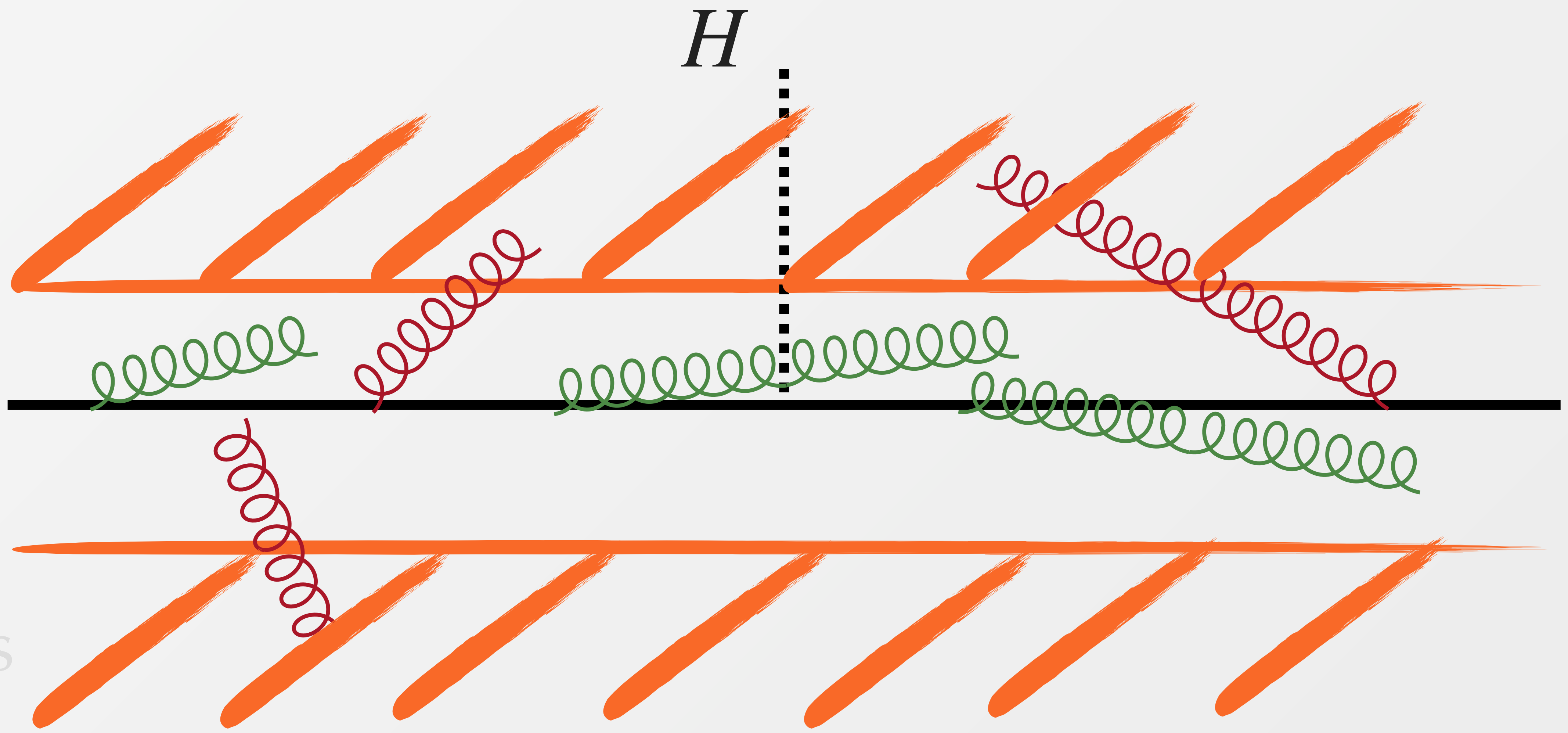

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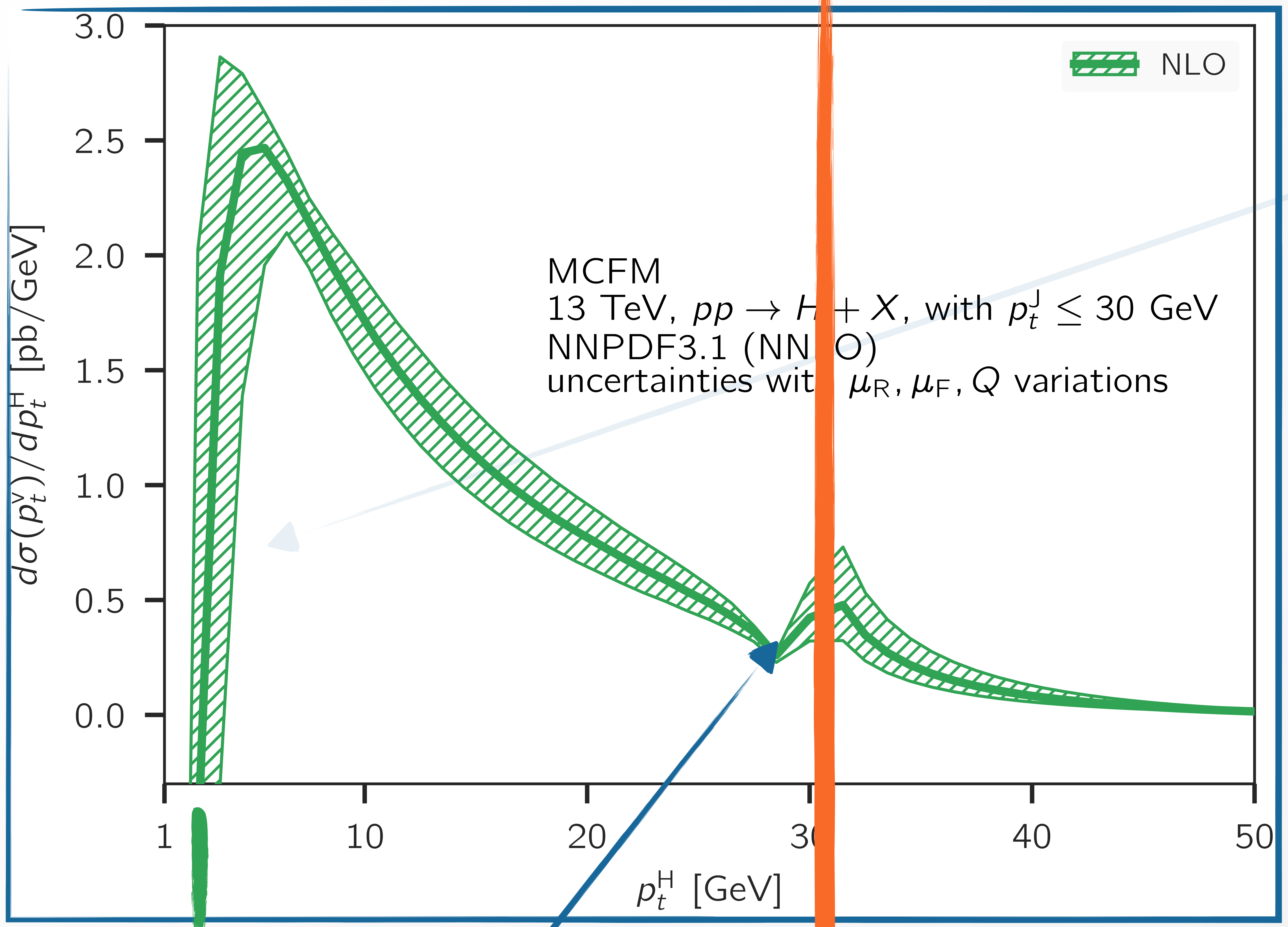

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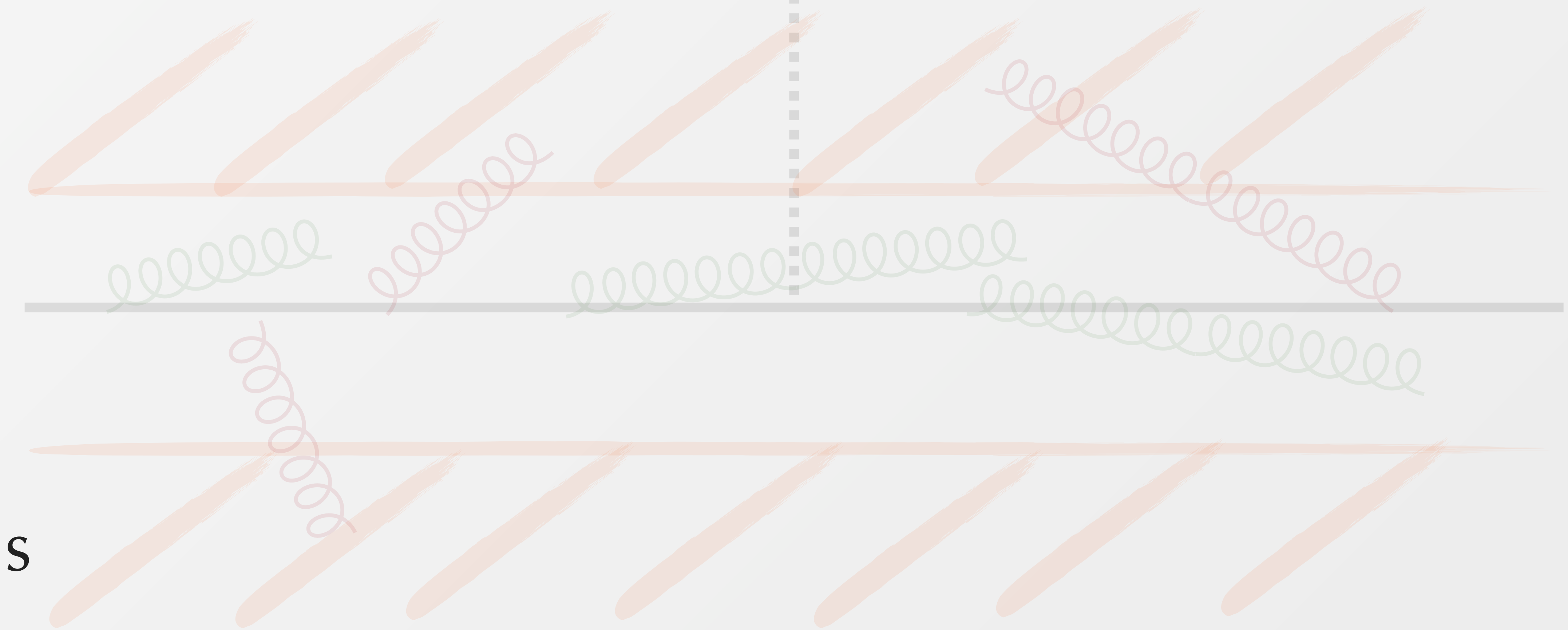

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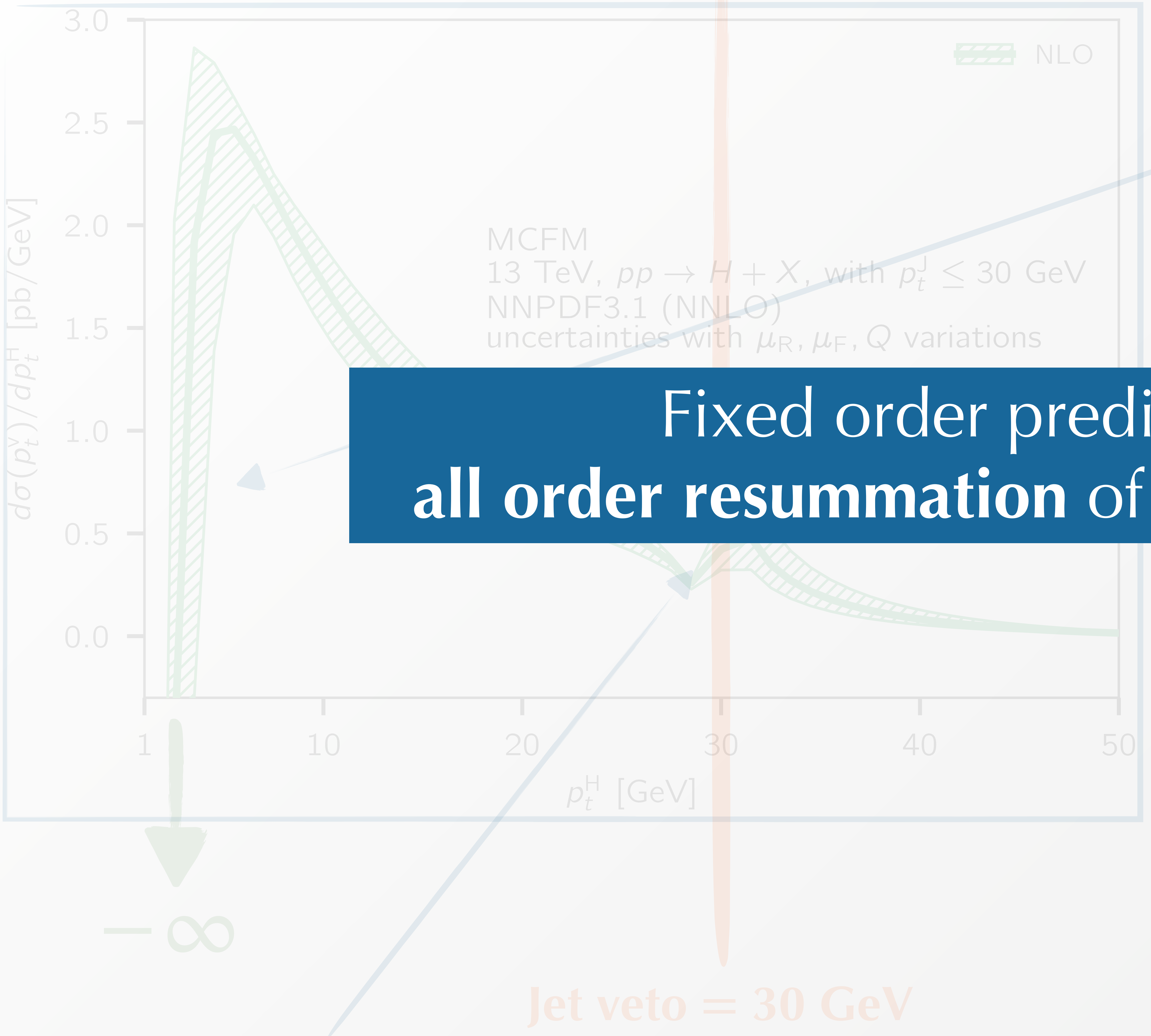

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Fixed order predictions no longer reliable:

all order resummation of the perturbative series mandatory

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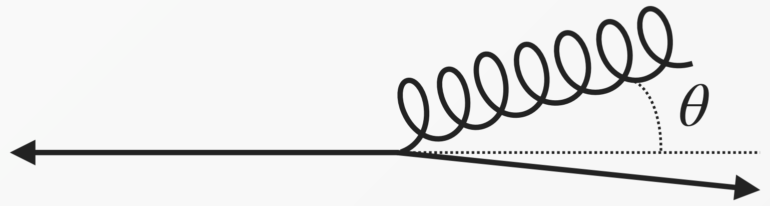
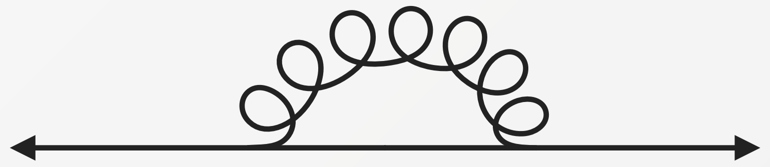
It’s not a bug, it’s a feature

Real emission diagrams singular for **soft/collinear emission**. Singularities are cancelled by virtual counterparts for IRC safe observables

Consider processes where real radiation is **constrained** in a corner of the phase space, (exclusive boundary of the phase space, **restrictive cuts**)

$$\tilde{\sigma}_1(p_\perp) \sim \underbrace{\int \frac{d\theta}{\theta}\frac{dE}{E}\Theta\left(p_\perp - E\theta\right)} - \underbrace{\int \frac{d\theta}{\theta}\frac{dE}{E}}$$





$$\sim -\int \frac{dE}{E}\frac{d\theta}{\theta}\Theta(E\theta - p_\perp)$$

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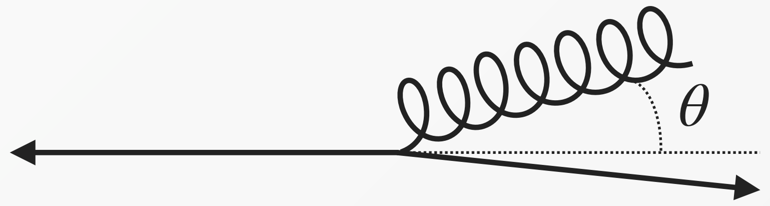
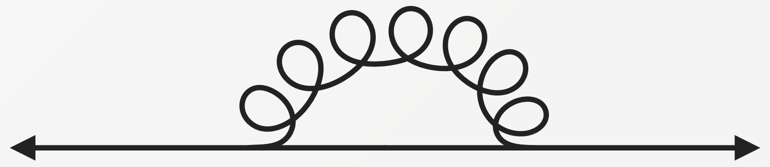




$p_\perp \to 0$: observable can become negative even in the perturbative regime

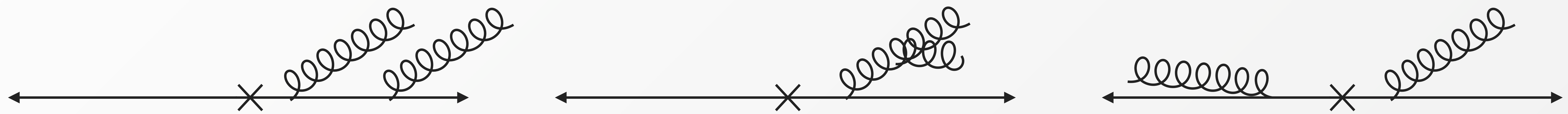
$$\sim -\int \frac{dE}{E}\frac{d\theta}{\theta}\Theta(E\theta - p_\perp) \sim -\frac{1}{2}\ln^2 p_\perp/m_H$$

Sudakov logarithms

Double logarithms **leftovers** of the real-virtual cancellation of IRC divergences

Making pQCD great again: all-order resummation

Soft-collinear emission of two gluons



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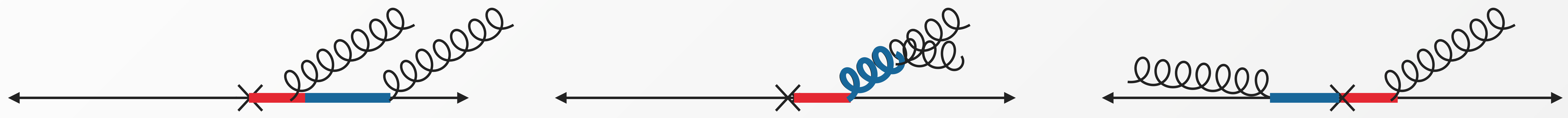

Two propagators nearly on shell, 4 divergences. Diagrams can potentially give $\alpha_s^2 \ln^4 p_\perp / m_H$

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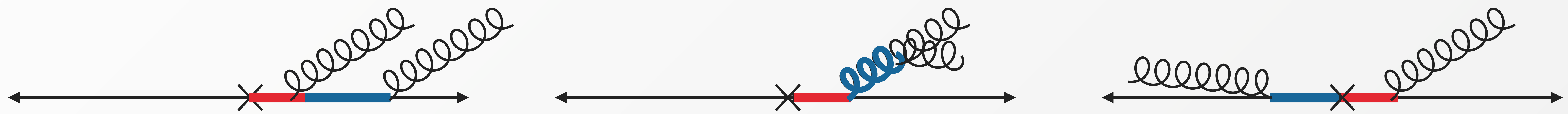

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All order structure

$$\tilde{\sigma}(v) = \sum_{n=0}^{\infty} \alpha_s^n \sum_{m=1}^{2n} c_{nm} L^m + \ldots \qquad L = \ln(p_\perp/m_H)$$

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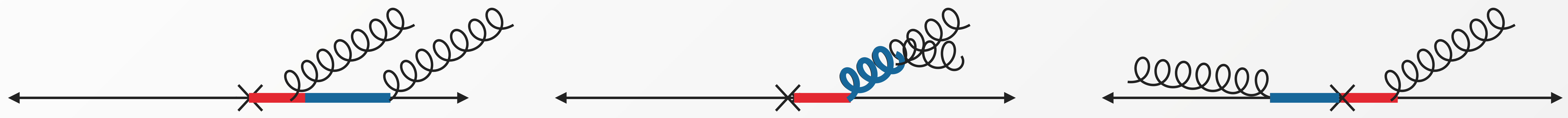

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Origin of the logs is simple. Resum them to all orders by **reorganizing** the series

$$\tilde{\sigma}(v) = f_1(\alpha_s L^2) + \frac{1}{L} f_2(\alpha_s L^2) + \dots$$

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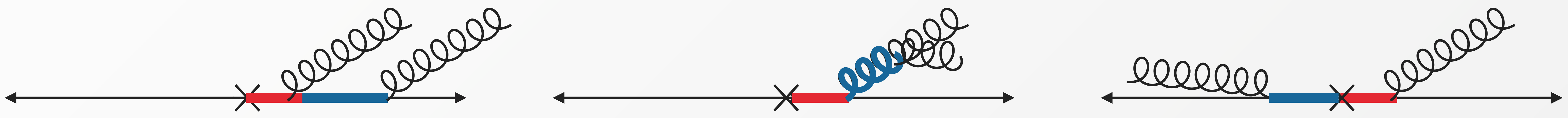

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Poor man's leading logarithmic (LL) resummation of the perturbative series

Accurate for $L \sim 1/\sqrt{\alpha_s}$

All-order resummation: exponentiation

Independent emissions $k_1, \ldots k_n$ (plus corresponding virtual contributions) in the soft and collinear limit with strong angular ordering

$$d\Phi_n |\mathcal{M}(k_1, \ldots k_n)|^2 \to \frac{1}{n!} \alpha_s^n \prod_{i=1}^{n} \frac{dE_i}{E_i} \frac{d\theta_i}{\theta_i}$$

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Calculate observable with arbitrary number of emissions: **exponentiation**

$$\tilde{\sigma} \simeq \sum_{n=0}^{\infty} \frac{1}{n!}\alpha_s^n \prod_{i=1}^{n} \int \frac{dE_i}{E_i}\frac{d\theta_i}{\theta_i}[\Theta(p_\perp - E_i\theta_i) - 1] \simeq e^{-\alpha_s L^2}$$

Sudakov suppression [Sudakov '54]
Price for constraining real radiation

Exponentiated form allows for a **more powerful reorganization**

$$\tilde{\sigma} = \exp\left[\sum_n \left(\underbrace{\mathcal{O}(\alpha_s^n L^{n+1})}_{\textbf{LL}} + \underbrace{\mathcal{O}(\alpha_s^n L^n)}_{\textbf{NLL}} + \underbrace{\mathcal{O}(\alpha_s^n L^{n-1})}_{\textbf{NNLL}} + \ldots\right)\right]$$

Region of applicability now valid up to $L \sim 1/\alpha_s$, successive terms suppressed by $\mathcal{O}(\alpha_s)$

All-order resummation: exponentiation

Independent emissions $k_1, \dots k_n$ (plus corresponding virtual contributions) in the soft and collinear limit with strong angular ordering

Exponentiation in direct space generally not possible.
Phase-space constraints typically do not factorize in direct space

$$\tilde{\sigma}(v) \sim \int \prod_i^n [dk_i] \, |\mathcal{M}(k_1, \dots, k_n)|^2 \, \Theta_{\rm PS}(v - V(k_1, \dots k_n))$$

How to achieve resummation?

Region of applicability now valid up to $L \sim 1/\alpha_s$, successive terms suppressed by $\mathcal{O}(\alpha_s)$

All-order resummation: (re)-factorization

Solution 1: move to **conjugate space** where phase space factorization is manifest

Exponentiation in conjugate space; **inverse transform** to move back to direct space

Extremely successful approach

"direct QCD"

- Catani, Trentadue, Mangano, Marchesini, Webber, Nason, Dokshitzer…
- Collins, Soper, Sterman, Laenen, Magnea…

SCET

- Manohar, Bauer, Stewart, Becher, Neubert….

+ many others!

Emphasis on properties of QCD matrix elements and QCD radiation

Factorization properties in the singular region and associated RGEs (factorization → evolution → resummation)

SCET vs. dQCD **not an issue** [Sterman *et al.* '13, '14][Bonvini, Forte, Ghezzi, Ridolfi, LR '12, '13, '14][Becher, Neubert *et al.* '08, '11, 14]

Limitation: it is **process-dependent**, and must be performed manually and analytically **for each observable**
for some complex observable difficult/impossible to derive **factorization theorem**

All-order resummation: CAESAR/ARES approach

Solution 2: Translate the resummability into properties of the observable in the presence of multiple radiation: **recursive infrared and collinear (rIRC) safety**

[Banfi, Salam, Zanderighi '01, '03, '04]
[Banfi, McAslan, Monni, Zanderighi, El-Menoufi '14, '18]

Simple observable easy to calculate

$$\tilde{\sigma} \sim \int \frac{dv_1}{v_1} \Sigma_s(v_1) \mathcal{F}(v, v_1)$$

Transfer function relates the resummation of the full observable to the one of the simple observable.

i.e. conditional probability

Method entirely formulated in **direct space**

Approach recently formulated within SCET language [Bauer, Monni '18, '19 + ongoing work]

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Separation obtained by introducing a **resolution scale** $q_0 = \epsilon k_{t,1}$

$$\tilde{\sigma} \sim \int [dk_1]\, e^{-R(q_0)}$$

Unresolved emission can be treated as **totally unconstrained** $\longrightarrow$ **exponentiation**

$$\times |\mathcal{M}(k_1)|^2 \left(\sum_{m=0}^{\infty} \frac{1}{m!} \int \prod_{i=2}^{m+1} [dk_i] |\mathcal{M}(k_i)|^2 \Theta(V(k_i) - q_0) \Theta\left(v - V(k_1, \ldots, k_{m+1})\right) \right)$$

Resolved emission treated exclusively with **Monte Carlo methods.** Integral is finite, can be integrated in d=4 with a computer

k_t-ordering

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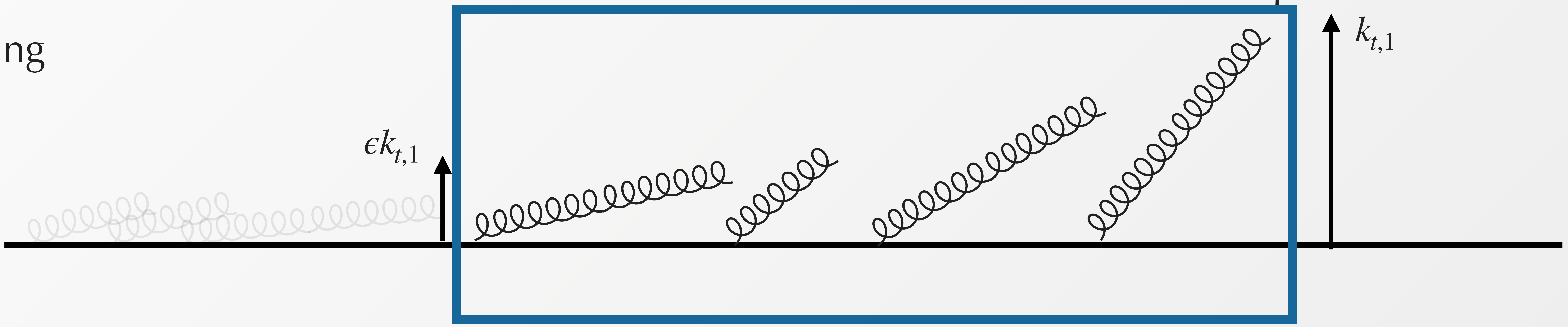
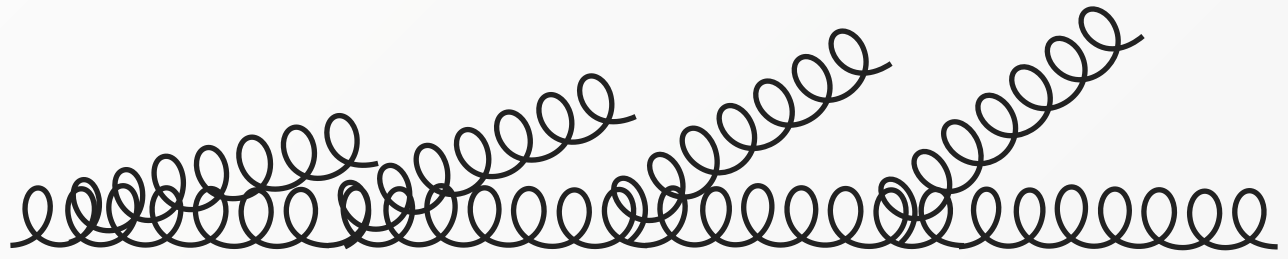


An example : resummation of the transverse momentum spectrum

Resummation of transverse momentum is particularly delicate because $p_\perp$ is a **vectorial quantity**

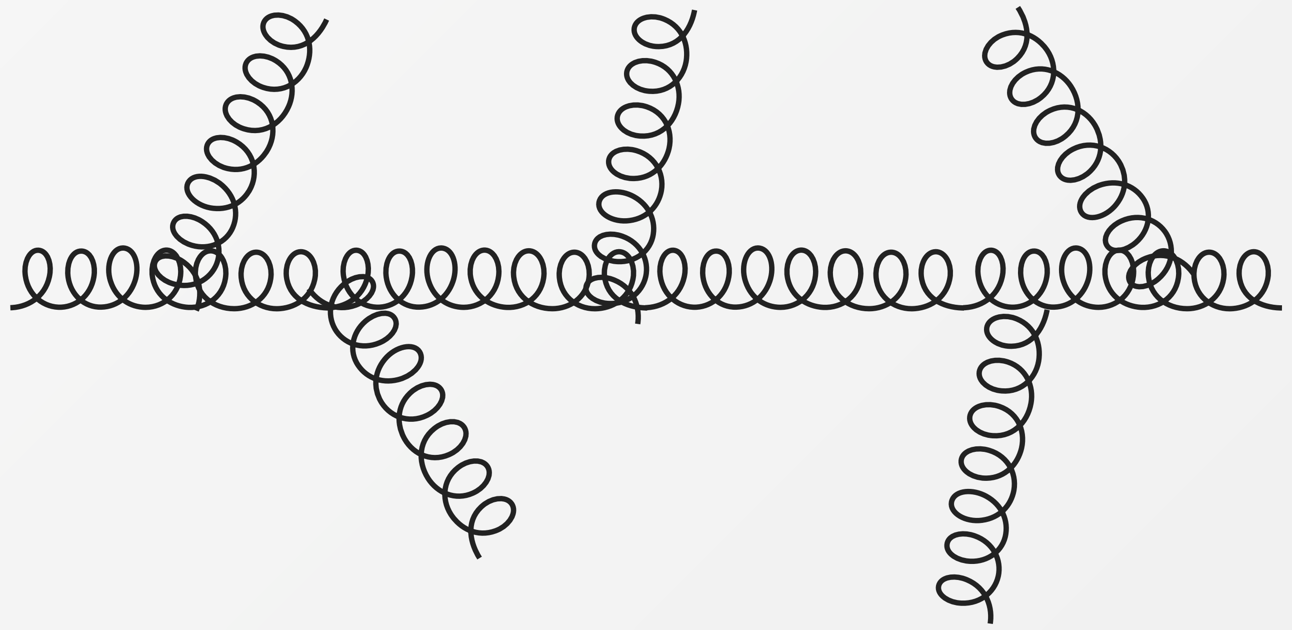
Two concurring mechanisms leading to a system with small $p_\perp$



$$p_\perp^2 \sim k_{t,i}^2 \ll m_H^2$$

cross section naturally suppressed as there is no phase space left for gluon emission (Sudakov limit)

Exponential suppression



$$\sum_{i=1}^{n} \vec{k}_{t,i} \simeq 0$$

Large kinematic cancellations

$p_\perp$ ~0 far from the Sudakov limit

Power suppression

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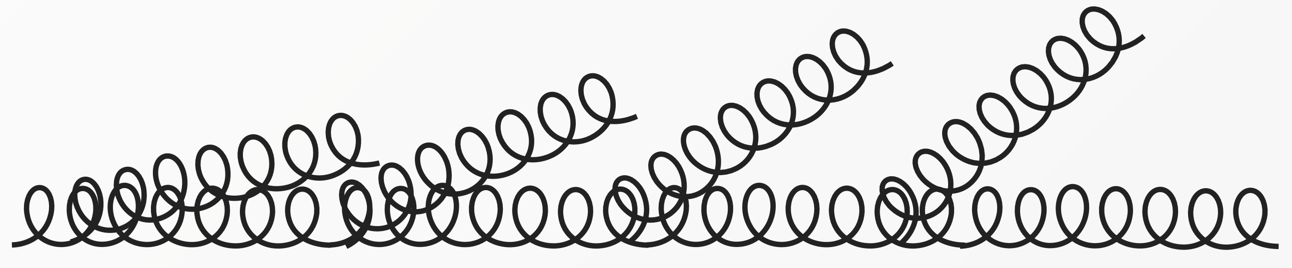

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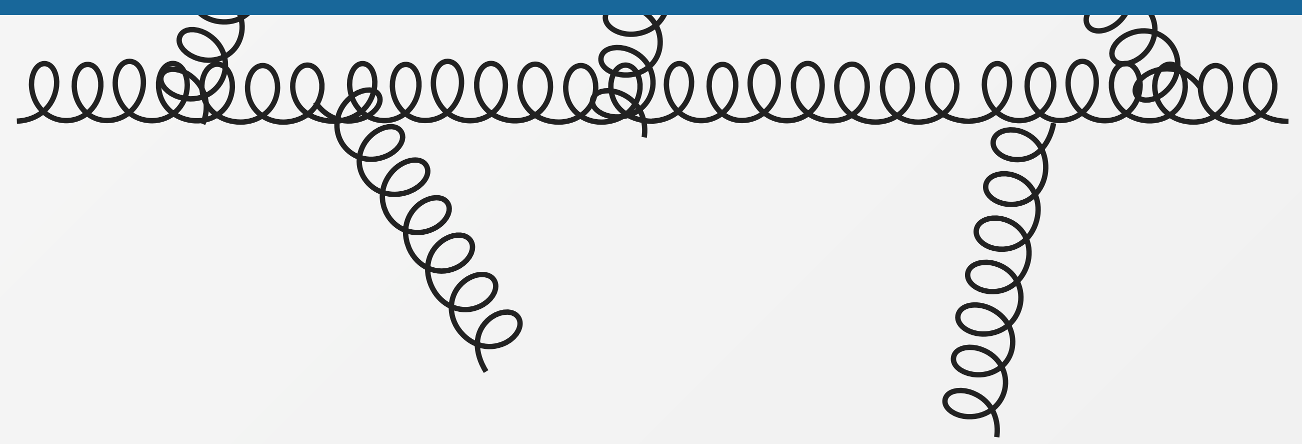
Exponential suppression

Dominant at small $p_\perp$



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Resummation of the transverse momentum spectrum in *b* space

Solution 1: move to **conjugate space** where phase space factorization is manifest

$p_\perp$ resummation
[Parisi, Petronzio '79; Collins, Soper, Sterman '85]

$$\delta^{(2)}\left(\vec{p}_t - \sum_{i=1}^{n} \vec{k}_{t,i}\right) = \int d^2b \frac{1}{4\pi^2} e^{i\vec{b}\cdot\vec{p}_t} \prod_{i=1}^{n} e^{-i\vec{b}\cdot\vec{k}_{t,i}}$$

two-dimensional momentum conservation

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two-dimensional momentum conservation

Exponentiation in conjugate space

virtual corrections

$$\sigma = \sigma_0 \int d^2\vec{p}_\perp^H \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p}_\perp^H} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int [dk_i] |M(k_i)|^2 \left(e^{i\vec{b}\cdot\vec{k}_{t,i}} - 1\right) = \sigma_0 \int d^2\vec{p}_\perp^H \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p}_\perp^H} e^{-R_{\rm NLL}(L)}$$

NLL formula with scale-independent PDFs

$$R_{\rm NLL}(L) = -Lg_1(\alpha_s L) - g_2(\alpha_s L) \qquad L = \ln(m_H b/b_0)$$

Logarithmic accuracy defined in terms of $\ln(m_H b/b_0)$

Resummation of the transverse momentum spectrum in direct space

Solution 2: Translate the resummability into properties of the observable in the presence of multiple radiation: **recursive infrared and collinear (rIRC) safety**

Resummation in direct space is a highly **non-trivial problem:** a naive resummation of logarithmic terms at small $p_\perp$ is not sensible, as one loses the **correct power-suppressed scaling** if only logarithms are retained.

It is not possible to reproduce a power-like behaviour with logs of $p_\perp/m_H$ [Frixione, Nason, Ridolfi '98]

Solution to the problem recently formulated by extending the CAESAR/ARES approach to deal with observables with azimuthal cancellations: **RadISH** approach [Monni, Re, Torrielli '16][Bizon, Monni, Re, LR, Torrielli '17]

Problem recently addressed also within SCET [Ebert, Tackmann '17]

Resummation of the transverse momentum spectrum in direct space

Result at NLL accuracy can be written as

$$\sigma(p_\perp) = \sigma_0 \int \frac{dv_1}{v_1} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(v_1)} \qquad v_i = k_{t,i}/m_H, \quad \zeta_i = v_i/v_1$$

$$\times\, \epsilon^{R'(v_1)} R'\left(v_1\right) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'\left(\zeta_i v_1\right)\, \Theta\left(p_\perp - |\vec{k}_{t,i} + \cdots \vec{k}_{t,n+1}|)\right)$$

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Simple observable

$$v_i = k_{t,i}/m_H, \quad \zeta_i = v_i/v_1$$

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Transfer function

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Transfer function

Formula can be evaluated with Monte Carlo method; dependence on ϵ vanishes (as $\mathcal{O}(\epsilon)$) and result is finite in four dimensions

Subleading effects retained: no divergence at small $p_\perp$, power-like behaviour respected

Logarithmic accuracy defined in terms of $\ln(m_H/k_{t1})$

Result formally equivalent to the b-space formulation [Bizon, Monni, Re, LR, Torrielli '17]

Direct space formulation

1. Similar in spirit to a **semi-inclusive parton shower**, but with higher-order logarithms, and **full control on the formal accuracy**

2. Thanks to its versatility, the approach can be exploited to formulate the resummation for entire classes of observables in an **unique framework**

3. **More differential description** of the QCD radiation than that usually possible in a conjugate-space formulation

Direct space formulation

Price to pay: less compact formulation

N³LL result

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1})\right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

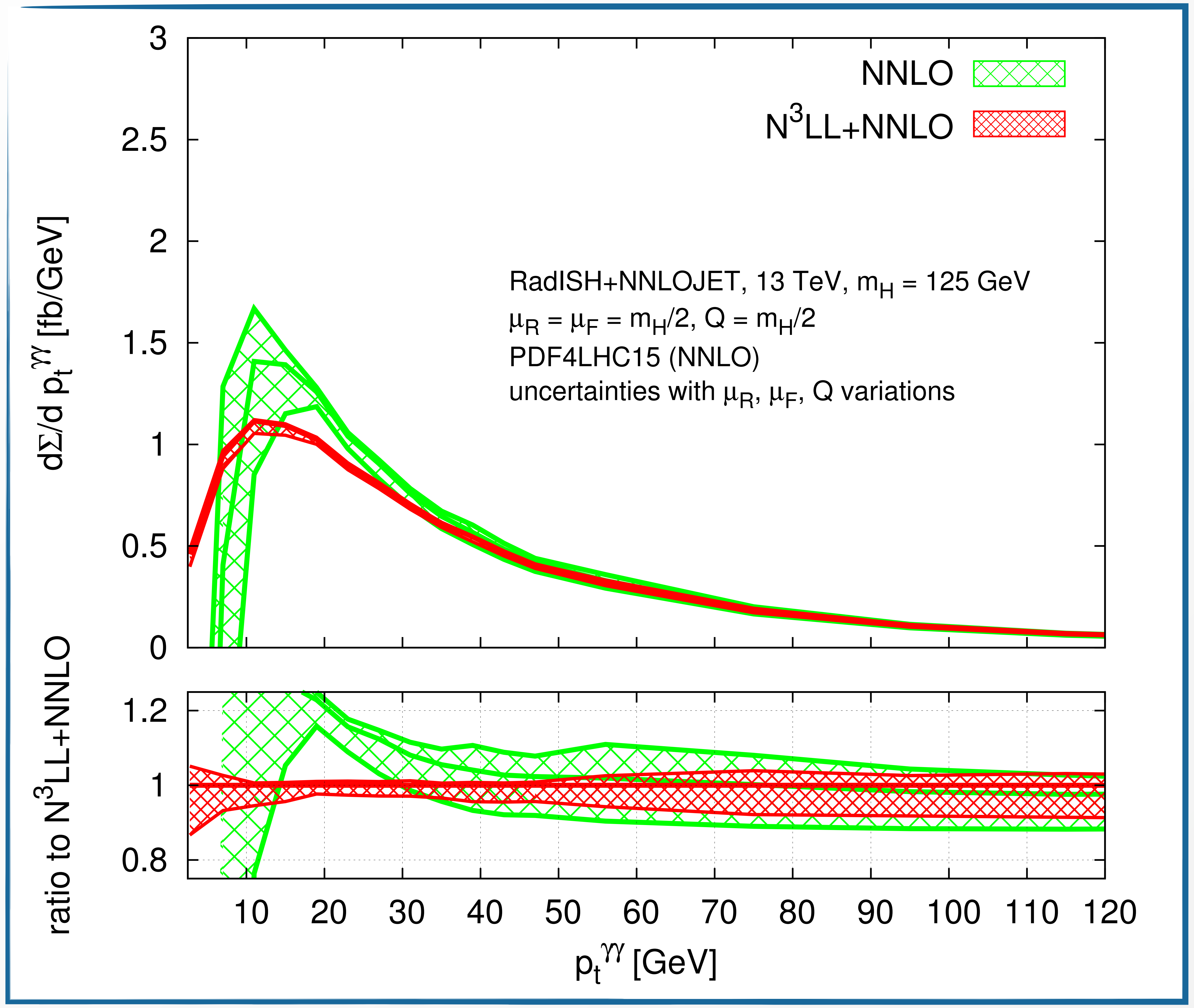
$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \left\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) \right) \right.$$
$$\times \left(R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2 \frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s} \right)$$
$$\left. + \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \right\} \left\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \right\}$$

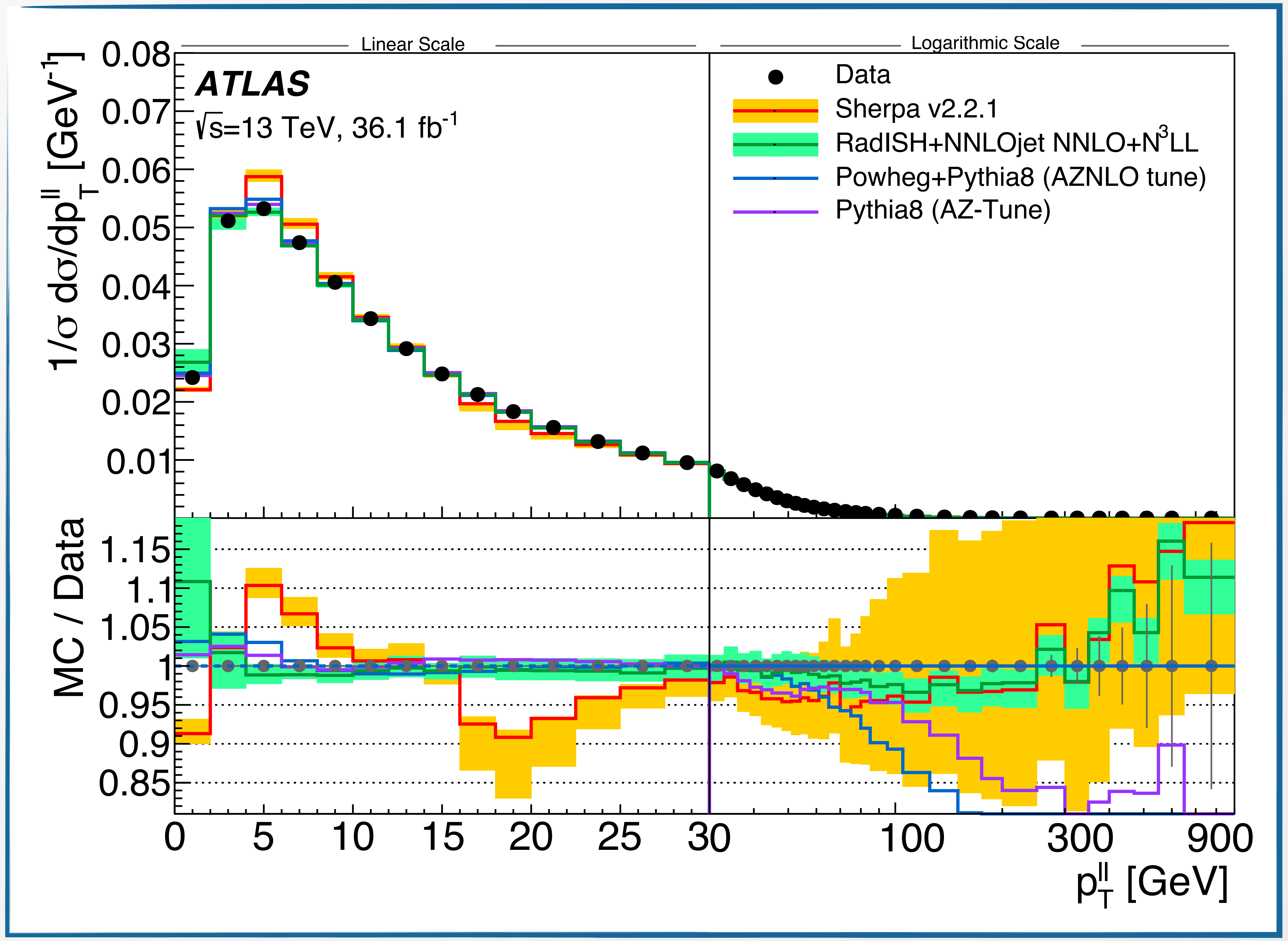
$$+ \frac{1}{2} \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}} \frac{d\phi_{s1}}{2\pi} \int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}} \frac{d\phi_{s2}}{2\pi} R'(k_{t1})$$
$$\times \left\{ \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \left(R''(k_{t1})\right)^2 \ln \frac{1}{\zeta_{s1}} \ln \frac{1}{\zeta_{s2}} - \partial_L \mathcal{L}_{\mathrm{NLL}}(k_{t1}) R''(k_{t1}) \left(\ln \frac{1}{\zeta_{s1}} + \ln \frac{1}{\zeta_{s2}} \right) \right.$$
$$\left. + \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \right\}$$
$$\times \left\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1}, k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1})\right) - \right.$$
$$\left. \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \right\} + \mathcal{O}\left(\alpha_s^n \ln^{2n-6} \frac{1}{v}\right), \qquad (3.18)$$

1. *Similar in spirit to a* ***semi-inclusive parton shower****, but with higher-order logarithms, and* ***full control on the formal accuracy***

Resummation of the transverse momentum spectrum at N³LL+NNLO

N³LL result matched to NNLO H+j, Z+j, W±+j [Bizon, LR *et al.* '17, '18, '19]





[ATLAS 1912.02844]

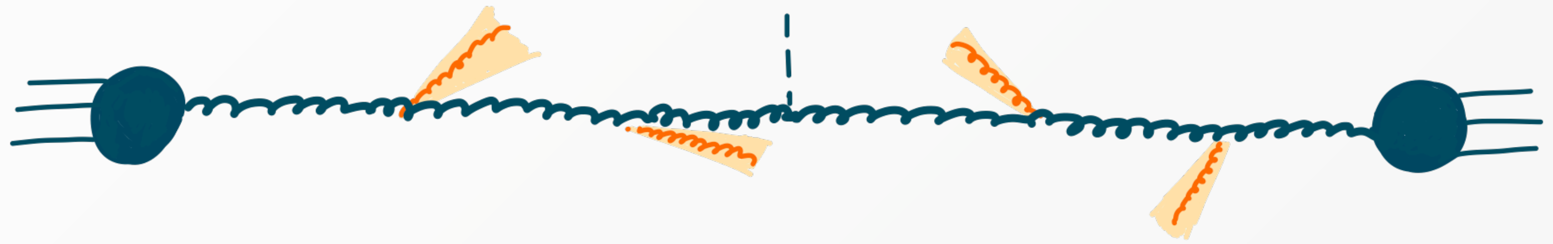
H+j at same accuracy also in SCET [Chen *et al.* '18]

2. *Thanks to its versatility, the approach can be exploited to formulate the resummation for entire classes of observables in an* ***unique framework***

Direct space formulation: generality

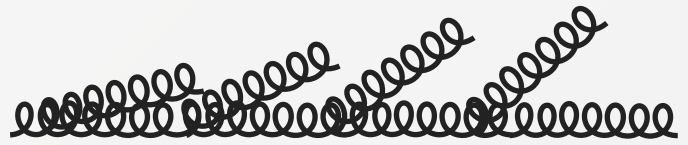
NLL result for $p_\perp^J$

$$\sigma(p_\perp^{\mathrm{J}}) = \sigma_0 e^{Lg_1(\alpha_s L) + g_2(\alpha_s L)}$$

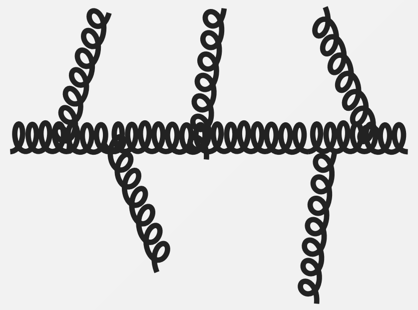


NLL result for $p_\perp^H$

$$\sigma(p_\perp^H) = \sigma_0 \int d^2\overrightarrow{p}_\perp^H \int \frac{d^2\overrightarrow{b}}{4\pi^2} e^{-i\overrightarrow{b}\cdot\overrightarrow{p}_\perp^H} e^{-R_{\mathrm{NLL}}(L)}$$



vs



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$$\sigma(p_\perp^{\rm J}) = \sigma_0 e^{Lg_1(\alpha_s L)+g_2(\alpha_s L)}$$

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General formula for a generic transverse observable at NLL [Bizon, Monni, Re, LR, Torrielli '17]

$$\sigma(v) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z}\, \Theta\left(v - V(k_1,\ldots,k_{n+1})\right)$$

$$d\mathcal{Z} = \epsilon^{R'(k_{t,1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'\left(\zeta_i k_{t,1}\right)$$

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General

differential control in momentum space provides guidance to double-differential resummation

$$\sigma(v) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z}\, \Theta\left(p_T^H - |\vec{k}_{t,1} + \cdots \vec{k}_{t,n+1}|\right)$$

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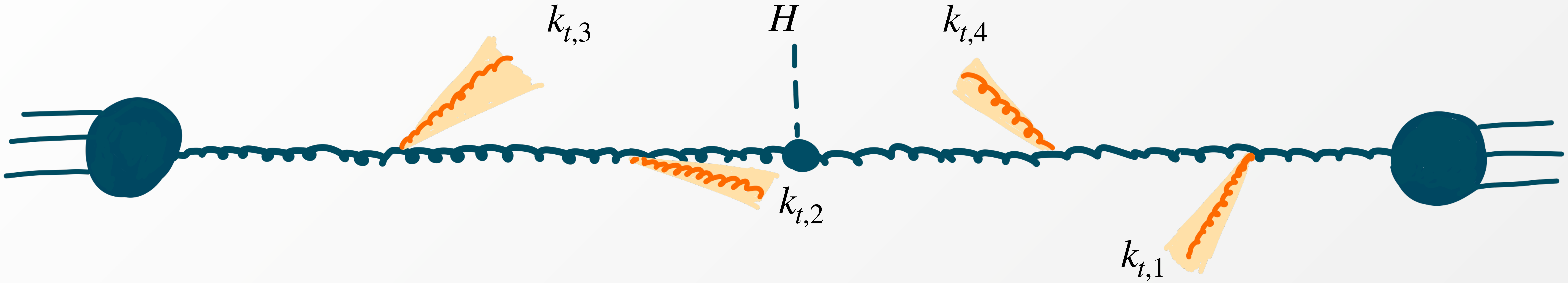
Double-differential resummation at NLL in b space

At NLL, emissions are **strongly ordered** in angle. k_t-clustering algorithms will associate **each emission** to a **different** jet



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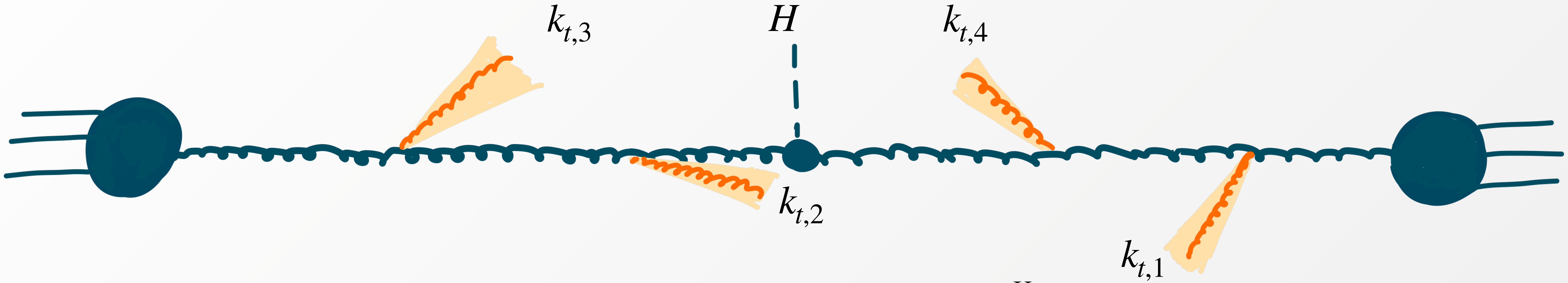

Additional constraint on **real radiation**

$$\Theta(p_\perp^{\mathrm{J,v}} - \max\{k_{t,1}, \ldots, k_{t,n}\}) = \prod_{i=1}^{n} \Theta(p_\perp^{\mathrm{J,v}} - k_{t,i})$$

$p_\perp^H$ resummation formula

$$\frac{d\sigma}{d^2\vec{p}_\perp^H} = \sigma_0 \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p}_\perp^H} e^{-R_{\mathrm{NLL}}(L)}$$

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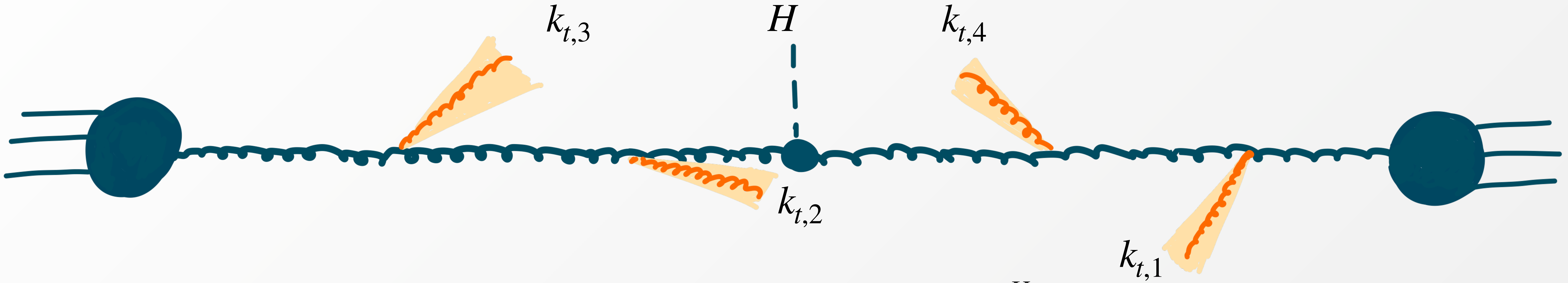

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Joint $p_\perp^H$, $p_\perp^{\mathrm{J,v}}$ resummation formula

$$\frac{d\sigma(p_\perp^{\mathrm{J,v}})}{d^2\vec{p}_\perp^H} = \sigma_0 \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p}_\perp^H} e^{-S_{\mathrm{NLL}}(L)}$$

CMW scheme [Catani, Marchesini, Webber '91]

$$S_{\mathrm{NLL}}(L) = -Lg_1(\alpha_s L) - g_2(\alpha_s L) + \int_0^{m_H} \frac{dk_t}{k_t} R'_{\mathrm{NLL}}(k_t) J_0(bk_t)\Theta(k_t - p_\perp^{\mathrm{J,v}})$$

$$R'_{\mathrm{NLL}}(k_t) = 4\left(\frac{\alpha_s^{\mathrm{CMW}}(k_t)}{\pi} C_A \ln\frac{m_H}{k_t} - \alpha_s(k_t)\beta_0\right)$$

Double-differential resummation at NNLL in b space

Crucial observation: in b space the phase space constraints entirely factorize → $e^{i\vec{b}\cdot\vec{k}_{t,i}}$

The jet veto constraint can be included by implementing the jet veto resummation at the b-space integrand level **directly in impact-parameter space**

Inclusive contribution: phase space constraint of the form

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Promote radiator at NNLL

$$S_{\mathrm{NNLL}} = -Lg_1(\alpha_s L) - g_2(\alpha_s L) - \alpha_s g_3(\alpha_s L) + \int_0^{m_H} \frac{dk_t}{k_t} R'_{\mathrm{NNLL}}(k_t) J_0(bk_t) \Theta(k_t - p_\perp^{\mathrm{J,v}})$$

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Resummation formula must be amended at NNLL [Banfi et al. '12][Becher et al. '13][Stewart et al. '14]

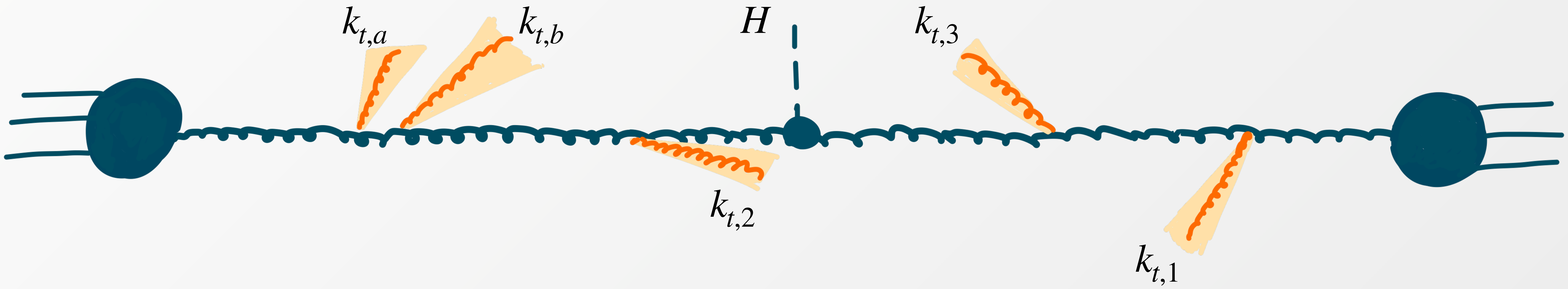
- **clustering correction**: jet algorithm can cluster two **independent** emissions into the same jet
- **correlated correction**: amends the inclusive treatment of the correlated squared amplitude for two emission accounting for configurations where the two **correlated** emissions (non abelian) are not clustered in the same jet

Double-differential resummation at NNLL in b space

clustering correction: jet algorithm can cluster two emissions into the same jet

$$\mathcal{F}_{\text{clust}} = \frac{1}{2!}\int [dk_a][dk_b]M^2(k_a)M^2(k_b)J_{ab}(R)\, e^{i\vec{b}\cdot\vec{k}_{t,ab}}\left[\Theta(p_\perp^{\text{J,v}} - k_{t,ab}) - \Theta(p_\perp^{\text{J,v}} - \max\{k_{t,a}, k_{t,b}\})\right]$$

$$J_{ab}(R) = \Theta\left(R^2 - \Delta\eta_{ab}^2 - \Delta\phi_{ab}^2\right)$$



Double-differential resummation at NNLL in b space

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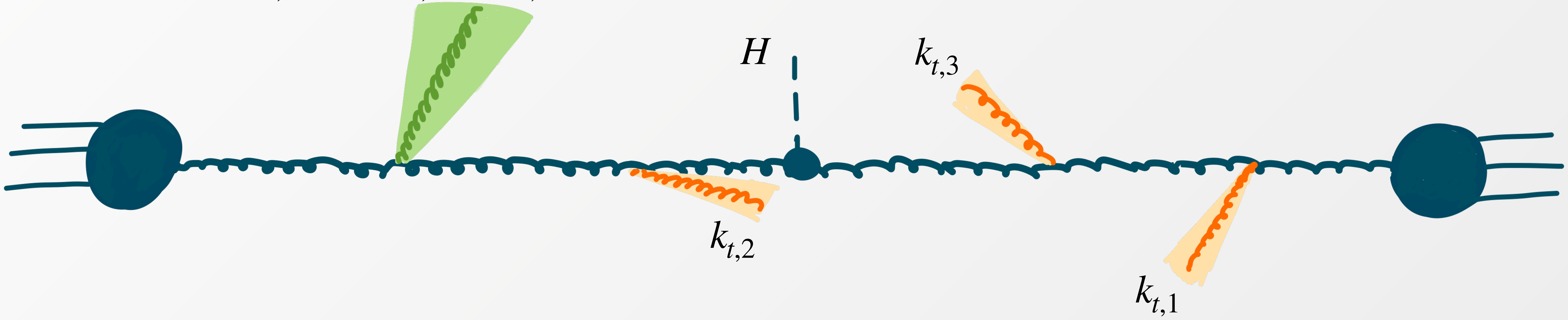
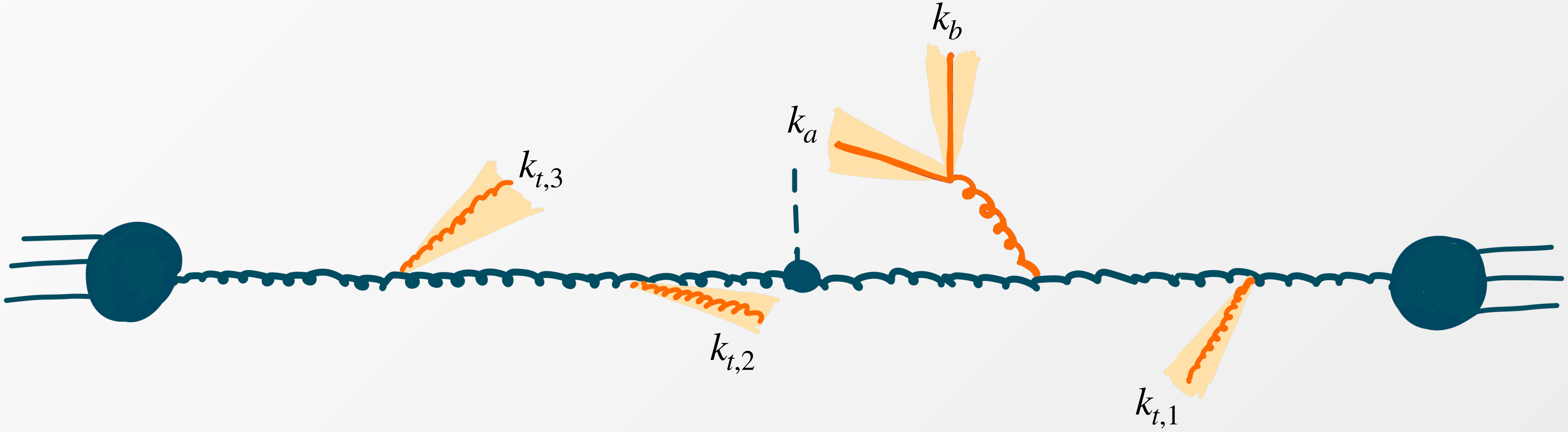
$$k_{t,ab} = |\vec{k}_{t,a} + \vec{k}_{t,b}|$$



Double-differential resummation at NNLL in b space

correlated correction: amends the inclusive treatment of the **correlated squared amplitude** for two emission accounting for configurations where the two correlated emissions are not clustered in the same jet

$$\mathcal{F}_{\rm correl} = \frac{1}{2!}\int [dk_a][dk_b]\tilde{M}^2(k_a,k_b)(1-J_{ab}(R))e^{i\vec{b}\cdot\vec{k}_{t,ab}} \times \left[\Theta(p_\perp^{\rm J,v} - \max\{k_{t,a},k_{t,b}\}) - \Theta(p_\perp^{\rm J,v} - k_{t,ab})\right]$$



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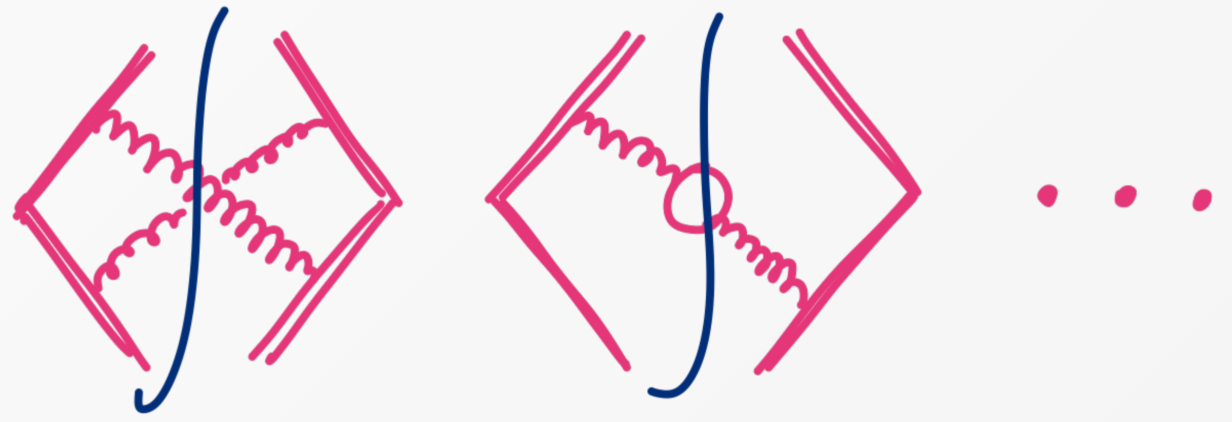

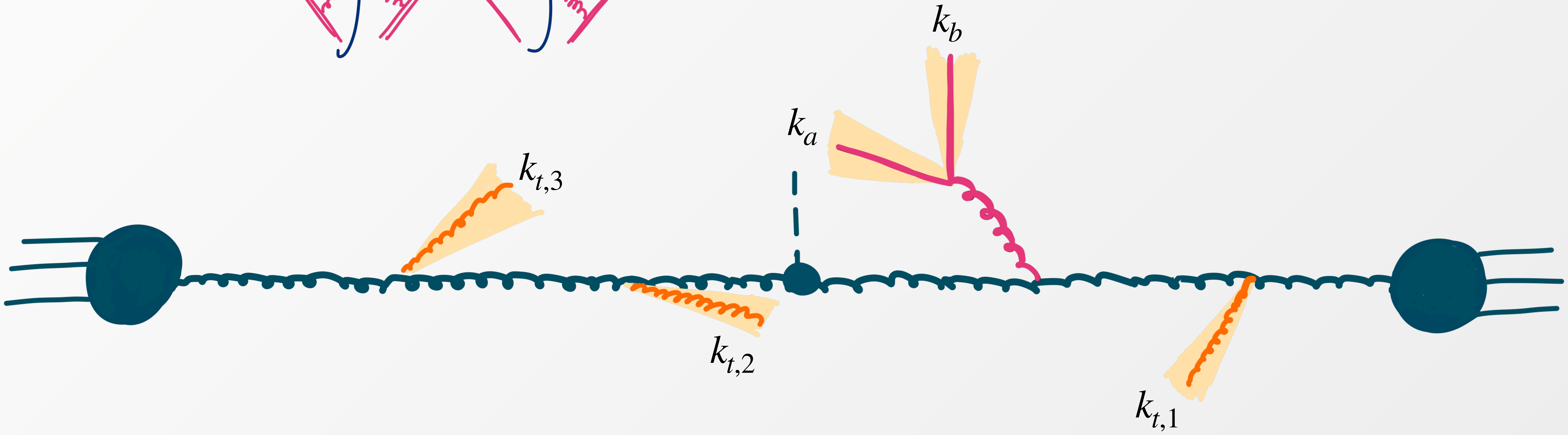

At NNLL, all remaining emissions can be considered to be far in angle from the pair k_a, k_b

Double-differential resummation at NNLL in b space

NNLL prediction finally requires the consistent treatment of non-soft collinear emissions off the initial state particles

Soft and non-soft emission cannot be clustered by a k_t-type jet algorithm. Non-soft collinear radiation can be handled by taking a Mellin transform of the resummed cross section, giving rise to **scale evolution of PDFs** and of the $\mathcal{O}(\alpha_s)$ **collinear coefficient functions**

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Soft and non-soft emission cannot be clustered by a k_t-type jet algorithm. Non-soft collinear radiation can be handled by taking a Mellin transform of the resummed cross section, giving rise to **scale evolution of PDFs** and of the $\mathcal{O}(\alpha_s)$ **collinear coefficient functions**

Final result at NNLL, including **hard-virtual corrections** at $\mathcal{O}(\alpha_s)$

$$\frac{d\sigma(p_\perp^{\mathrm{J,v}})}{dy_H d^2\vec{p}_\perp^{\,H}} = M^2_{\mathrm{gg}\to\mathrm{H}}\,\mathcal{H}(\alpha_s(m_H)) \int_{\mathcal{C}_1}\frac{d\nu_1}{2\pi i}\int_{\mathcal{C}_2}\frac{d\nu_2}{2\pi i}\, x_1^{-\nu_1}x_2^{-\nu_2}\int\frac{d^2\vec{b}}{4\pi^2}\, e^{-i\vec{b}\cdot\vec{p}_\perp^{\,H}}\, e^{-S_{\mathrm{NNLL}}}\left(1+\mathcal{F}_{\mathrm{clust}}+\mathcal{F}_{\mathrm{correl}}\right)$$

$$\times f_{\nu_1,a_1}(b_0/b)\, f_{\nu_2,a_2}(b_0/b)\left[\mathcal{P}\, e^{-\int_{p_\perp^{\mathrm{J,v}}}^{m_H}\frac{d\mu}{\mu}\mathbf{\Gamma}_{\nu_1}(\alpha_s(\mu))J_0(b\mu)}\right]_{c_1 a_1}\left[\mathcal{P}\, e^{-\int_{p_\perp^{\mathrm{J,v}}}^{m_H}\frac{d\mu}{\mu}\mathbf{\Gamma}_{\nu_2}(\alpha_s(\mu))J_0(b\mu)}\right]_{c_2 a_2}$$

$$\times C_{\nu_1,gb_1}(\alpha_s(b_0/b))\, C_{\nu_2,gb_2}(\alpha_s(b_0/b))\left[\mathcal{P}\, e^{-\int_{p_\perp^{\mathrm{J,v}}}^{m_H}\frac{d\mu}{\mu}\mathbf{\Gamma}^{(C)}_{\nu_1}(\alpha_s(\mu))J_0(b\mu)}\right]_{c_1 b_1}\left[\mathcal{P}\, e^{-\int_{p_\perp^{\mathrm{J,v}}}^{m_H}\frac{d\mu}{\mu}\mathbf{\Gamma}^{(C)}_{\nu_2}(\alpha_s(\mu))J_0(b\mu)}\right]_{c_2 b_2}$$

Mellin moments

Flavour indices

Double-differential resummation in direct space

Just need to **combine measurement functions**!

At NLL

$$\sigma(p_\perp^H) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z} \Theta\left(p_\perp^H - |\vec{k}_{t,1} + \cdots \vec{k}_{t,n+1}|\right)$$

Double-differential resummation in direct space

Just need to **combine measurement functions**!

At NLL

$$\sigma(p_\perp^{\mathrm{J,v}}) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z} \Theta\left(p_\perp^{\mathrm{J,v}} - \max\left\{k_{t,1}, \ldots k_{t,n+1}\right\}\right)$$

Double-differential resummation in direct space

Just need to **combine measurement functions**!

At NLL

$$\sigma(p_\perp^{\mathrm{J,v}}, p_\perp^{H}) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z} \Theta\left(p_\perp^{\mathrm{J,v}} - \max\left\{k_{t,1}, \ldots k_{t,n+1}\right\}\right) \Theta\left(p_\perp^{H} - |\vec{k}_{t,1} + \cdots \vec{k}_{t,n+1}|\right)$$

Double-differential resumation in direct space

☐ Just need to **combine measurement functions**!

At NLL

$$\sigma(p_\perp^{\mathrm{J,v}}, p_\perp^{H}) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z}\, \Theta\left(p_\perp^{\mathrm{J,v}} - \max\left\{k_{t,1}, \ldots k_{t,n+1}\right\}\right) \Theta\left(p_\perp^{H} - |\vec{k}_{t,1} + \cdots \vec{k}_{t,n+1}|\right)$$

Same philosophy at NNLL

$$\sigma^{\mathrm{NNLL}}(p_\perp^{\mathrm{J,v}}) = \sigma^{\mathrm{NNLL}}_{\mathrm{incl}}(p_\perp^{\mathrm{J,v}}) + \sigma^{\mathrm{NNLL}}_{\mathrm{clust}}(p_\perp^{\mathrm{J,v}}) + \sigma^{\mathrm{NNLL}}_{\mathrm{corr}}(p_\perp^{\mathrm{J,v}})$$

where e.g.

$$\sigma^{\mathrm{NNLL}}_{\mathrm{clust}}(p_\perp^{\mathrm{J,v}}) \simeq \int_0^\infty \frac{dk_{t,1}}{k_{t,1}} \frac{d\phi_1}{2\pi} \int d\mathcal{Z}\, e^{-R(k_{t,1})}\, 8\, C_A^2 \frac{\alpha_s^2(k_{t,1})}{\pi^2}\, \Theta\left(p_\perp^{\mathrm{J,v}} - \max_{i>1}\{k_{t,i}\}\right)$$
$$\times \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{d\phi_{s_1}}{2\pi} \int_{-\infty}^{\infty} d\Delta\eta_{1s_1} J_{1s_1}(R) \left[\Theta\left(p_\perp^{\mathrm{J,v}} - |\vec{k}_{t,1} + \vec{k}_{t,s_1}|\right) - \Theta\left(p_\perp^{\mathrm{J,v}} - k_{t,1}\right)\right]$$

Double-differential resummation in direct space

[?] Just need to **combine measurement functions**!

At NLL

$$\sigma(p_\perp^{\mathrm{J,v}}, p_\perp^{H}) = \sigma_0 \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} R'\left(k_{t,1}\right) d\mathcal{Z}\, \Theta\left(p_\perp^{J,v} - \max\left\{k_{t,1}, \ldots k_{t,n+1}\right\}\right) \Theta\left(p_\perp^{H} - |\vec{k}_{t,1} + \cdots \vec{k}_{t,n+1}|\right)$$

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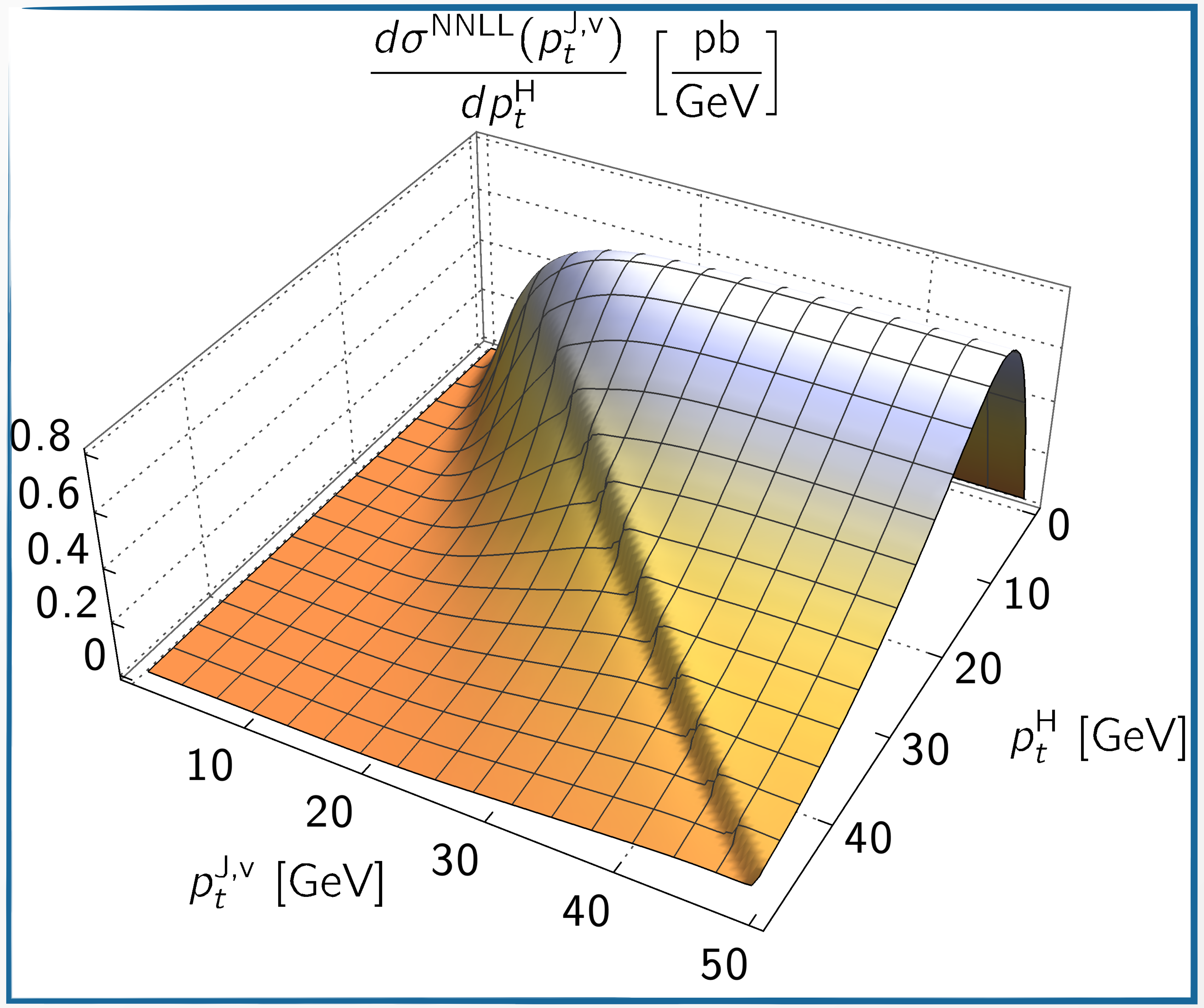
$$\times \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{d\phi_{s_1}}{2\pi} \int_{-\infty}^{\infty} d\Delta\eta_{1s_1} J_{1s_1}(R) \left[\Theta\left(p_\perp^{\mathrm{J,v}} - |\vec{k}_{t,1} + \vec{k}_{t,s_1}|\right) - \Theta\left(p_\perp^{\mathrm{J,v}} - k_{t,1}\right)\right]$$
$$\times\, \Theta\left(p_\perp^{H} - |\vec{k}_{t,1} + \ldots + \vec{k}_{t,n+1} + \vec{k}_{t,s_1}|\right)$$

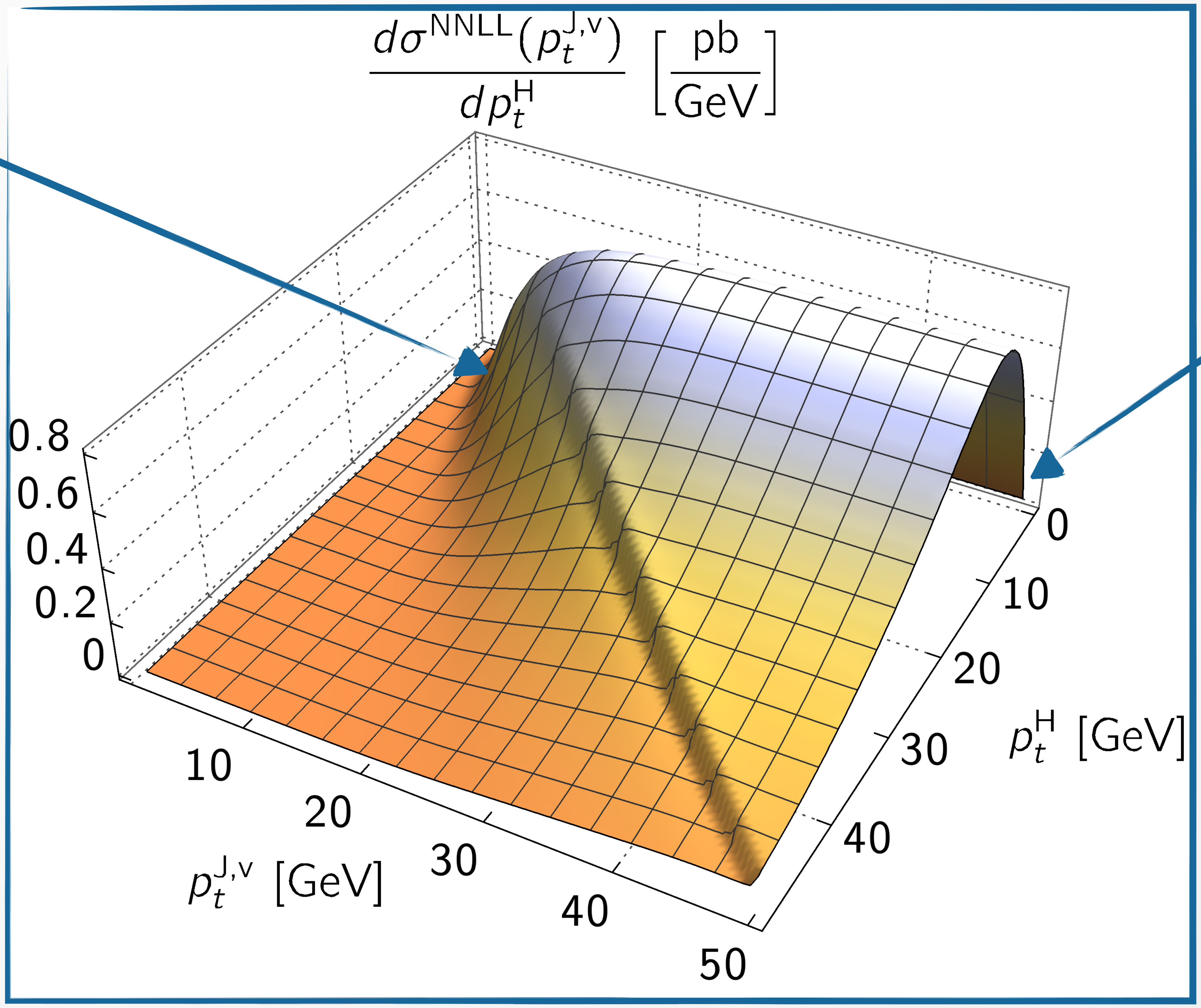
And analogously for other contributions

NNLL cross section differential in $p_\perp^H$, cumulative in $p_\perp^J \leq p_\perp^{J,v}$



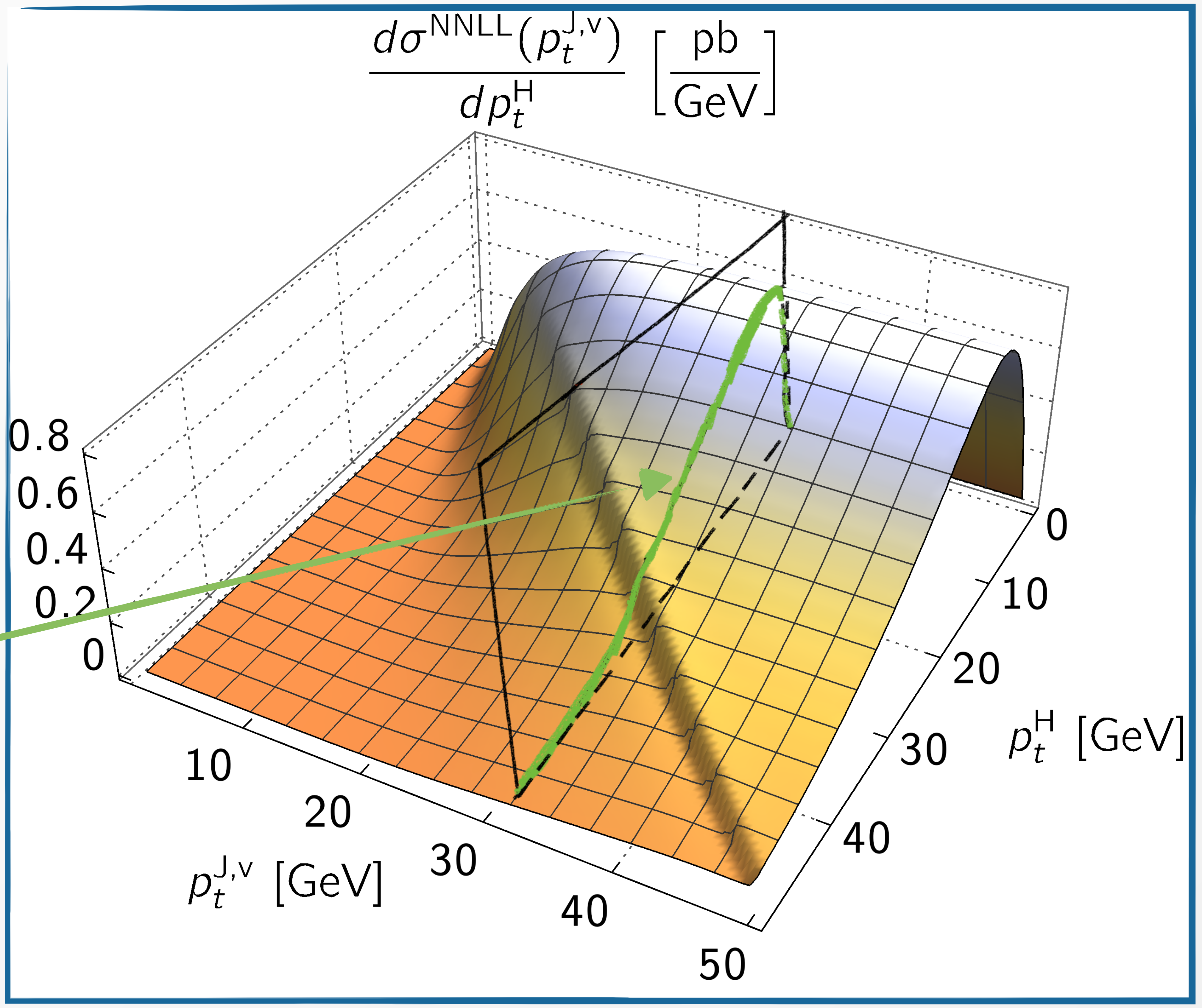
NNLL cross section differential in $p_\perp^H$, cumulative in $p_\perp^J \leq p_\perp^{J,v}$

Sudakov suppression at small $p_\perp^J$

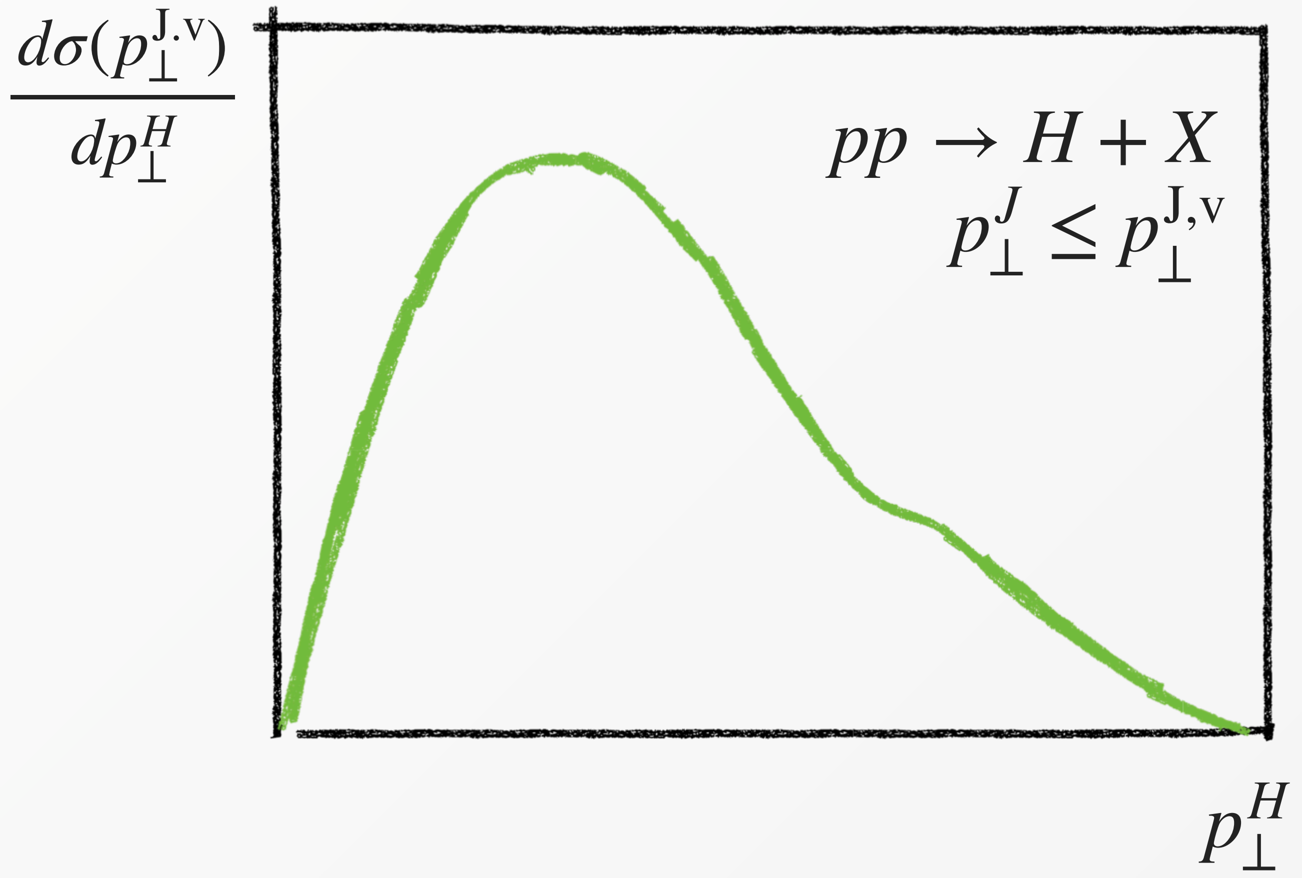


Peaked structure (Sudakov) + power-like suppression at very small $p_\perp^H$

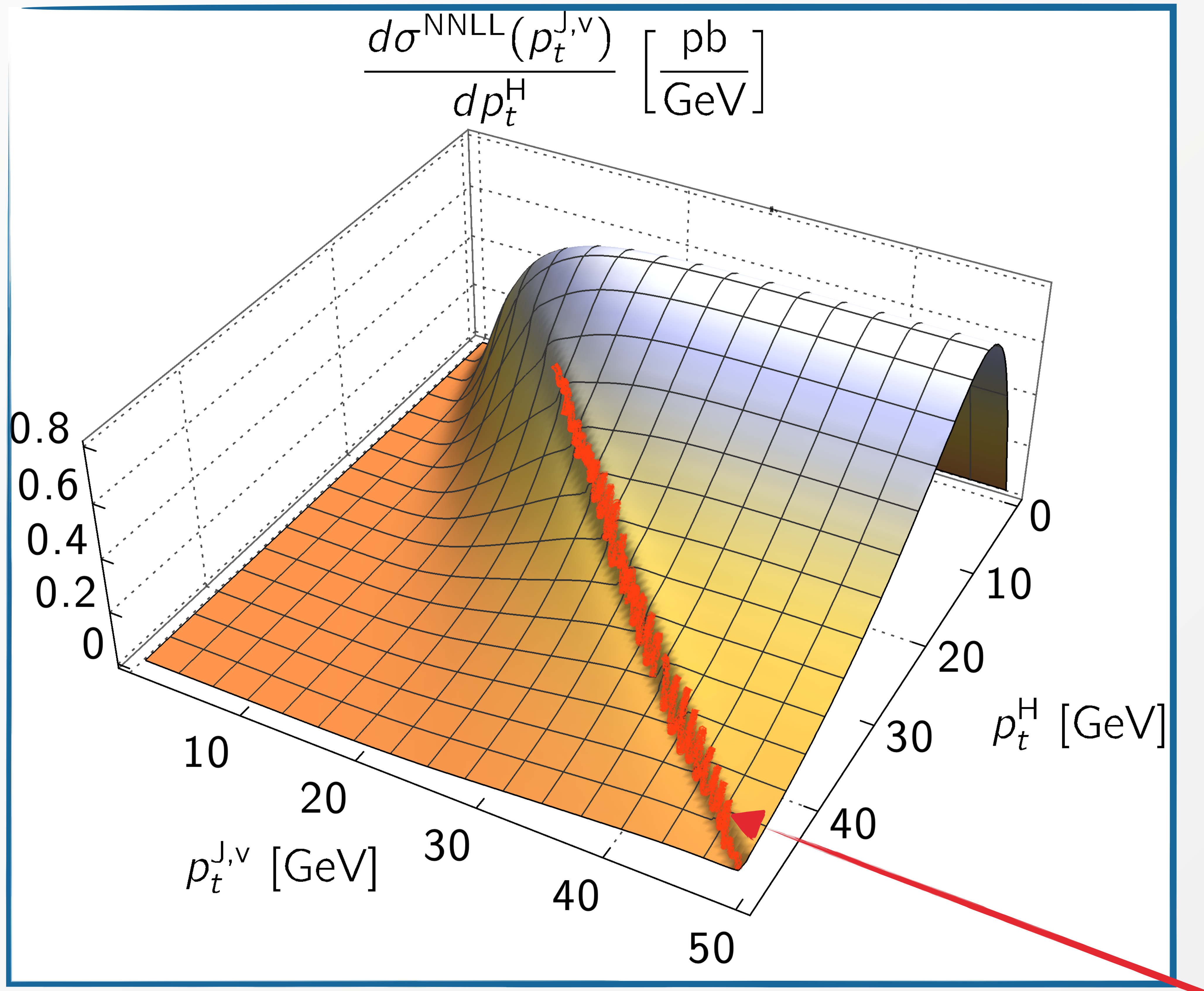
NNLL cross section differential in $p_\perp^H$, cumulative in $p_\perp^{\mathrm{J}} \le p_\perp^{\mathrm{J,v}}$



At a given value of $p_\perp^{\mathrm{J,v}}$ it corresponds to the $p_\perp^H$ cross section in the 0-jet bin



NNLL cross section differential in $p_\perp^H$, cumulative in $p_\perp^J \leq p_\perp^{J,v}$



[Catani, Webber '97]

Sudakov shoulder: integrable singularity beyond LO at $p_\perp^H \simeq p_\perp^{J,v}$

Logarithms associated to the Shoulder are resummed in the limit $p_\perp^H \sim p_\perp^{J,v} \ll m_H$

Accuracy check at $\mathcal{O}(\alpha_s^2)$

Comparison of the expansion of the resummed result with the fixed order at $\mathcal{O}(\alpha_s^2)$ in the limit $p_\perp^H \sim p_\perp^{\mathrm{J,v}} \ll m_H$

$$\sigma^{\mathrm{NNLO}}(p_\perp^H < p_\perp^{H,\mathrm{v}}, p_\perp^{\mathrm{J}} < p_\perp^{\mathrm{J,v}}) = \sigma^{\mathrm{NNLO}} - \int \Theta(p_\perp^H > p_\perp^{H,\mathrm{v}}) \vee \Theta(p_\perp^{\mathrm{J}} > p_\perp^{\mathrm{J,v}}) d\sigma_{H+\mathrm{J}}^{\mathrm{NLO}}$$

$$\sim \sigma_0(1 + \alpha_s(L^2 + L + c_1) + \alpha_s^2(L^4 + L^3 + L^2 + L + c_1))$$

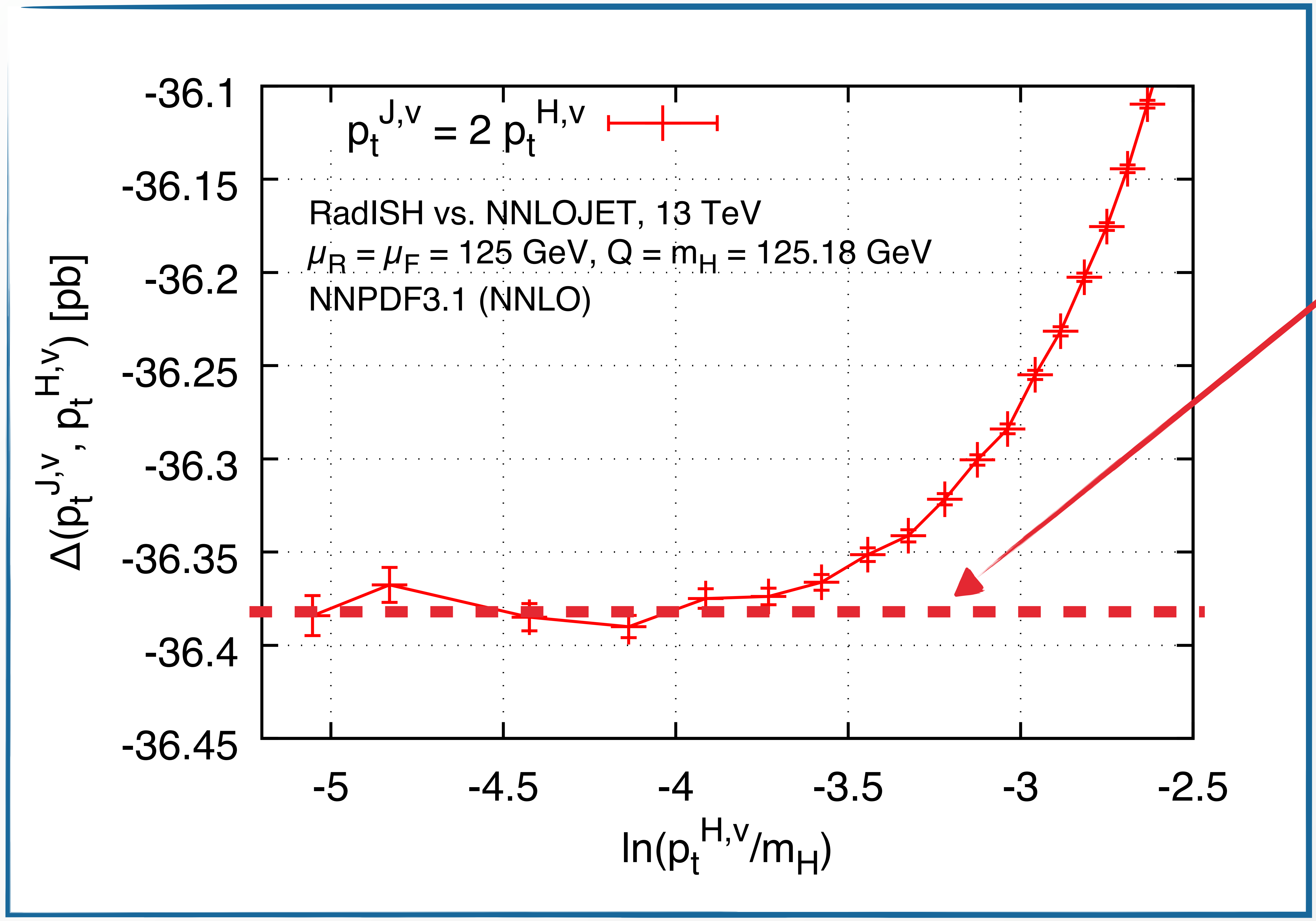
Fully predicted at NNLL

Difference at the double-cumulative level

$$\Delta(p_\perp^{\mathrm{J,v}}, p_\perp^{H,\mathrm{v}}) = \sigma^{\mathrm{NNLO}}(p_\perp^{\mathrm{J,v}}, p_\perp^{H,\mathrm{v}}) - \sigma_{\mathrm{exp.}}^{\mathrm{NNLL}}(p_\perp^{\mathrm{J,v}}, p_\perp^{H,\mathrm{v}})$$

should go to a **constant** when the logarithms get large ($p_\perp^H \sim p_\perp^{\mathrm{J,v}} \ll m_H$)

Accuracy check at $\mathcal{O}(\alpha_s^2)$



Difference at the double-cumulative level goes to a **constant** (all logarithmic terms correctly predicted)

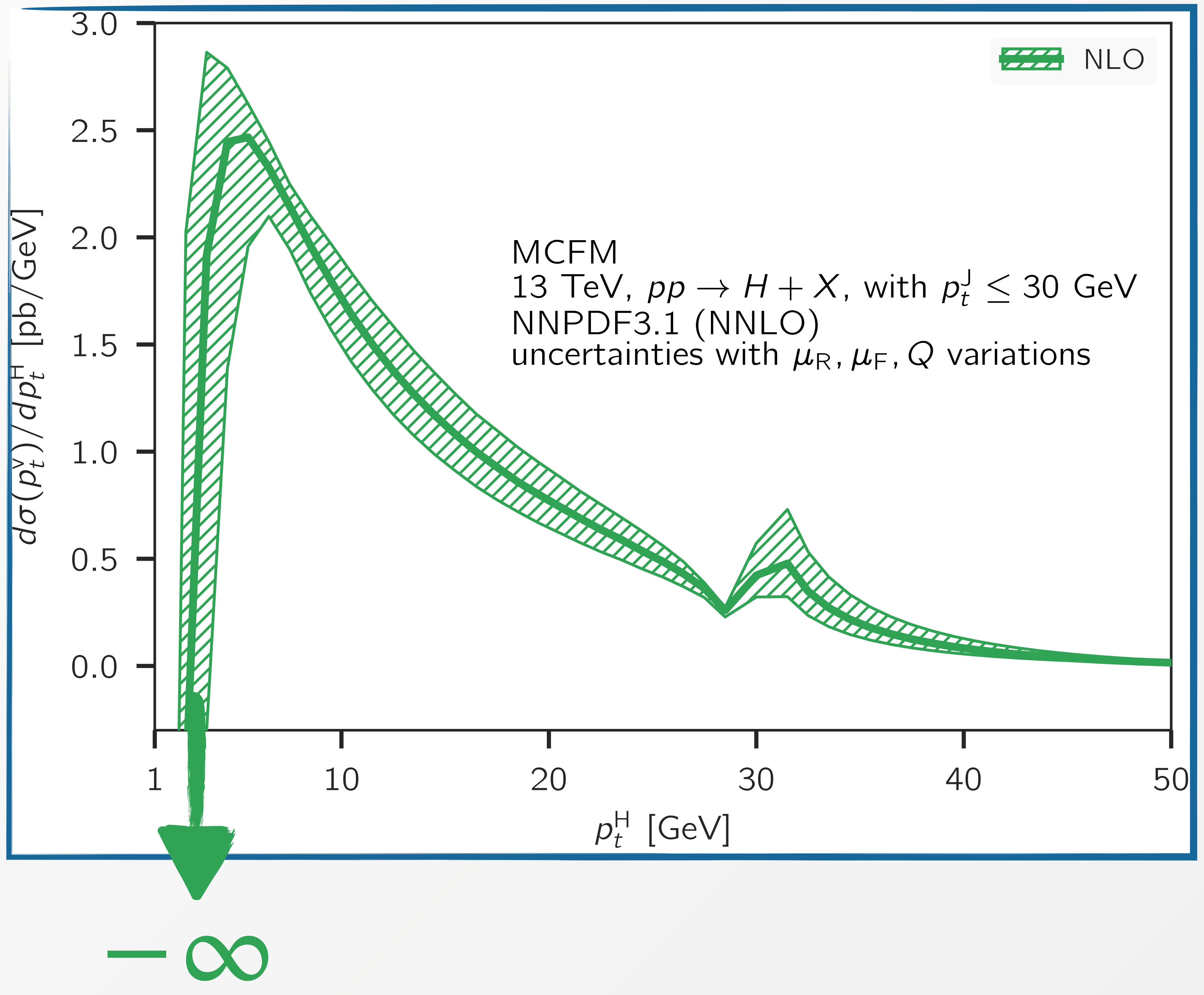
Very strong check: **NNLL resummation** of the logarithms associated to the shoulder

Analogous checks performed in the limits $p_\perp^H \ll p_\perp^{\mathrm{J,v}} < m_H$ and $p_\perp^{\mathrm{J,v}} \ll p_\perp^H < m_H$

LHC results: Higgs transverse momentum with a jet veto

Multiplicative matching to fixed order (NLO H+j from MCFM, NNLO H from ggHiggs)

[Campbell, Ellis, Giele,'15] [Bonvini et al '13]



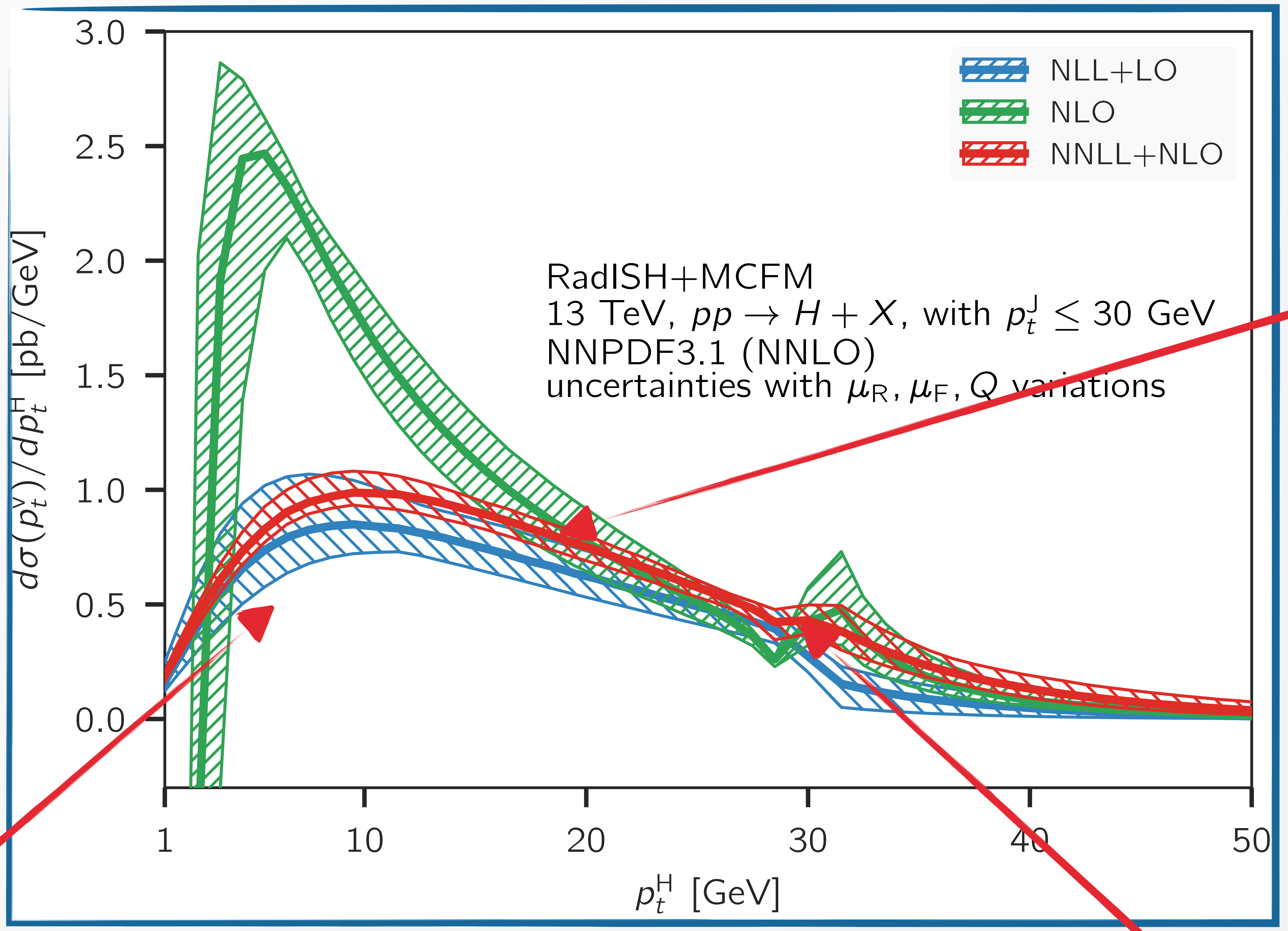
LHC results: Higgs transverse momentum with a jet veto

Multiplicative matching to fixed order (NLO H+j from MCFM, NNLO H from ggHiggs)

[Campbell, Ellis, Giele,'15] [Bonvini et al '13]

residual uncertainties at **NNLL+NLO** at the 10% level

large K-factor becomes relevant at larger $p_\perp^H$



much reduced sensitivity to the Sudakov shoulder with respect to NLO spectrum

good perturbative convergence to the left of the shoulder; above, multi-particle configurations play a substantial role

LHC applications: W^+W^- production

Jet vetoed analyses commonly enforced in LHC searches

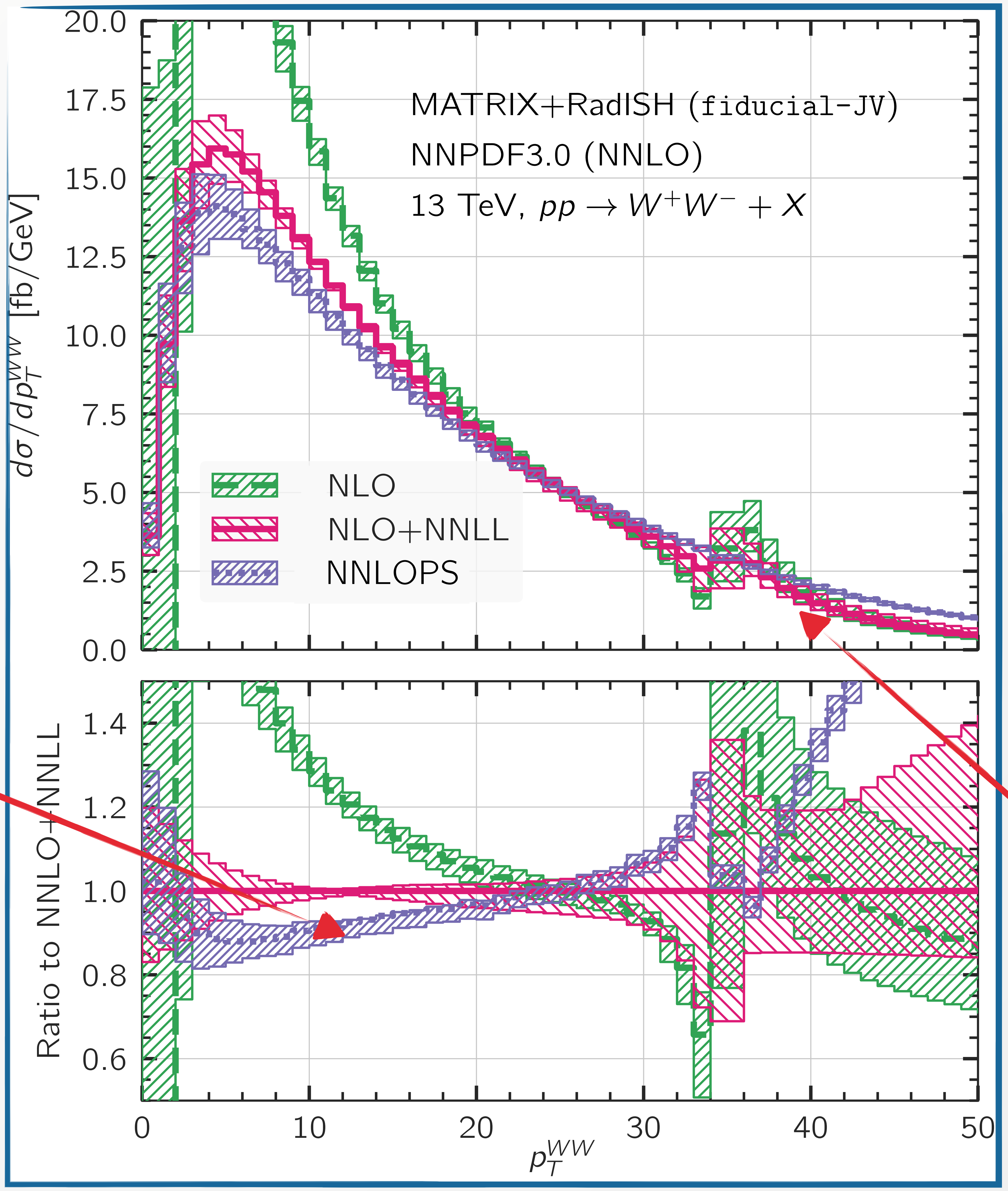
For instance, W^+W^- channel, which is relevant for BSM searches into leptons missing energy and/or jets and Higgs measurements, suffers from a signal contamination due to large top-quark background

Fiducial region defined by a rather stringent jet veto

W^+W^- transverse momentum with a jet veto

RadISH+MATRIX interface for generic 2⟶1 and 2⟶2 colour singlet processes

[Kallweit, Wiesemann, Re, LR 2004.07720]



Fixed order predictions from MATRIX

[Grazzini, Kallweit, Rathlev, Wiesemann '15, '17]

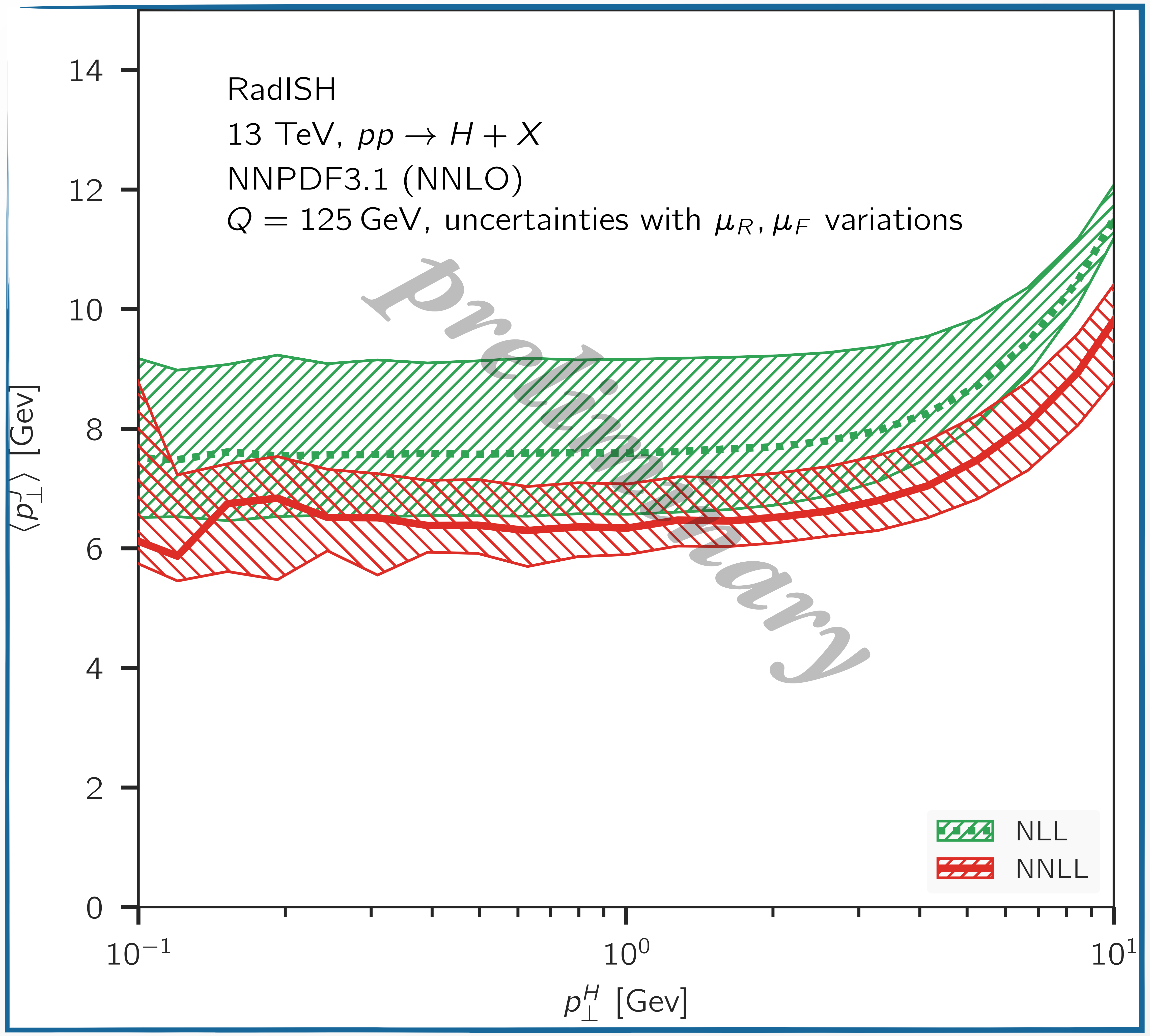
[Wiesemann, Re, Zanderighi '18]

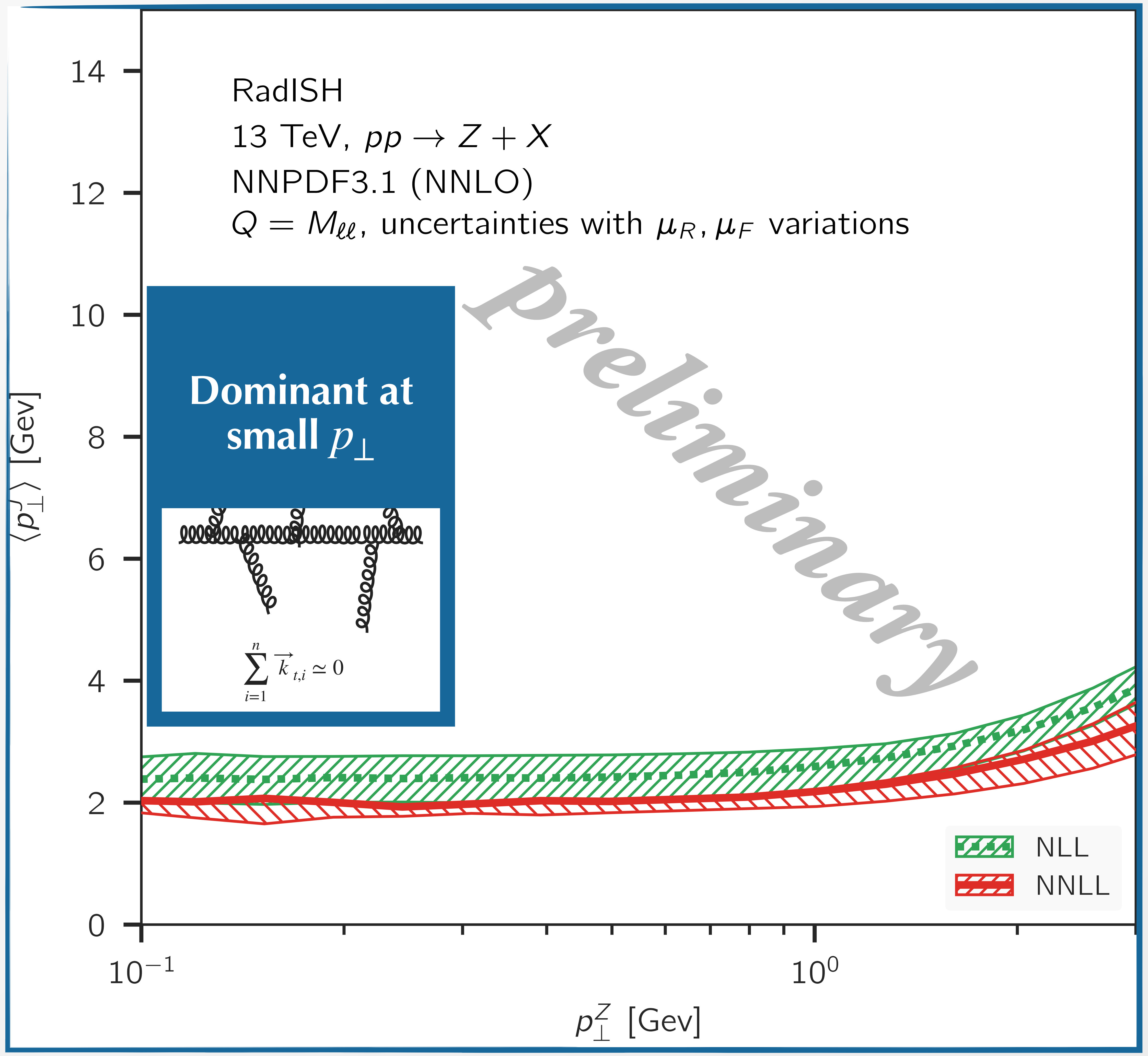
Comparison with NNLOPS result (much lower log accuracy) shows differences at the $\mathcal{O}(10\%)$ level

reduced sensitivity to the Sudakov shoulder with respect to NLO spectrum

*3. **More differential description** of the QCD radiation than that usually possible in a conjugate-space formulation*

Direct space: access to differential information and underlying dynamics





Possible access to subleading jets and higher moments

Summary

- Precision of the data demands an increasing theoretical accuracy at the **multi-differential level** to fully exploit LHC potential
- First **joint resummation** for a **double-differential** kinematic observable involving a **jet algorithm** in hadronic collisions
- Direct space formulation (RadISH) provides guidance to obtain **elegant and compact formulation in *b*-space** at NNLL accuracy and offers access to underlying dynamics
- Formalism can be readily extended to **more complex final states**; $2\longrightarrow 1$ and $2\longrightarrow 2$ colour singlet processes soon available via upcoming MATRIX+RadISH framework

Backup

All-order structure of the matrix element

$v = p_t/M$

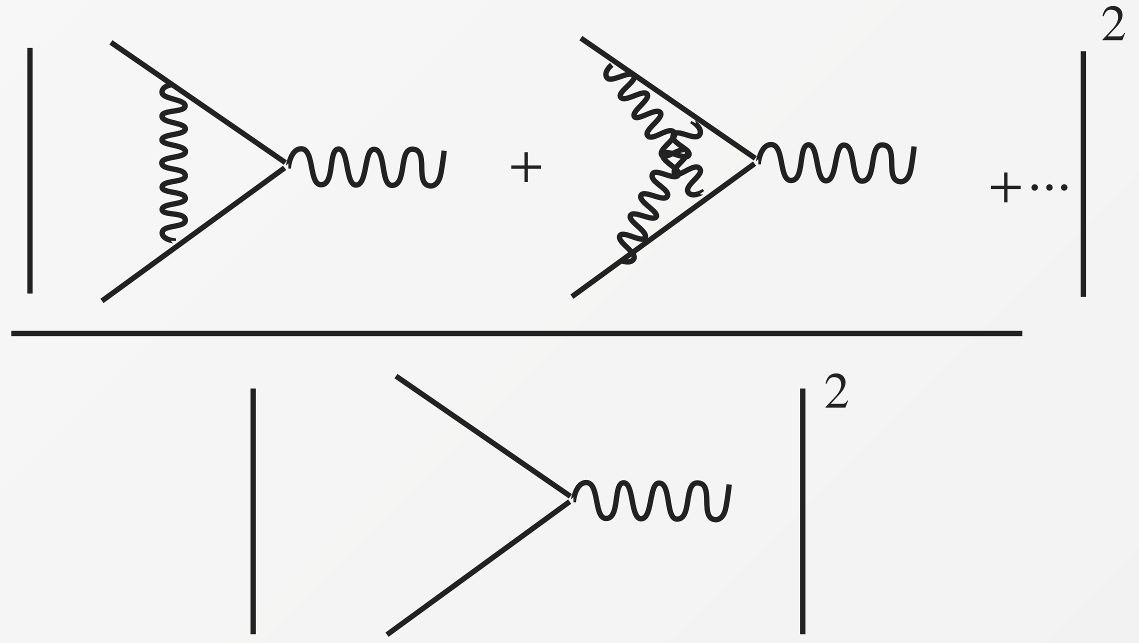
single-particle phase space

matrix element for *n* real emissions

$$\Sigma(v) = \int d\Phi_B \mathscr{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] \, |\mathscr{M}(\Phi_B, k_1, \ldots k_n)|^2 \Theta(v - V(\{\Phi_B\}, k_1, \ldots k_n))$$

all-order form factor (virtuals)

e.g. [Dixon, Magnea, Sterman '08]



Transverse observable resummation with RadISH

1. Establish a **logarithmic counting** for the squared matrix element $|\mathcal{M}(\Phi_B, k_1, \ldots k_n)|^2$

Decompose the squared amplitude in terms of ***n*-particle correlated blocks,** denoted by $|\tilde{\mathcal{M}}(k_1, \ldots, k_n)|^2$
$(|\tilde{\mathcal{M}}(k_1)|^2 = |\mathcal{M}(k_1)|^2)$

$$\sum_{n=0}^{\infty} |\mathcal{M}(\Phi_B, k_1, \ldots, k_n)|^2 = |\mathcal{M}_B(\Phi_B|^2$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left(|\mathcal{M}(k_i)|^2 + \int [dk_a][dk_b] |\tilde{\mathcal{M}}(k_a, k_b)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} - \vec{k}_{ti}) \delta(Y_{ab} - Y_i) \right. \right.$$

$$\left. \left. + \int [dk_a][dk_b][dk_c] |\tilde{\mathcal{M}}(k_a, k_b, k_c)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} + \vec{k}_{tc} - \vec{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \right) \right\} \equiv |\mathcal{M}_B(\Phi_B)|^2 \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} |\mathcal{M}(k_i)|^2_{\text{inc}}$$

LL

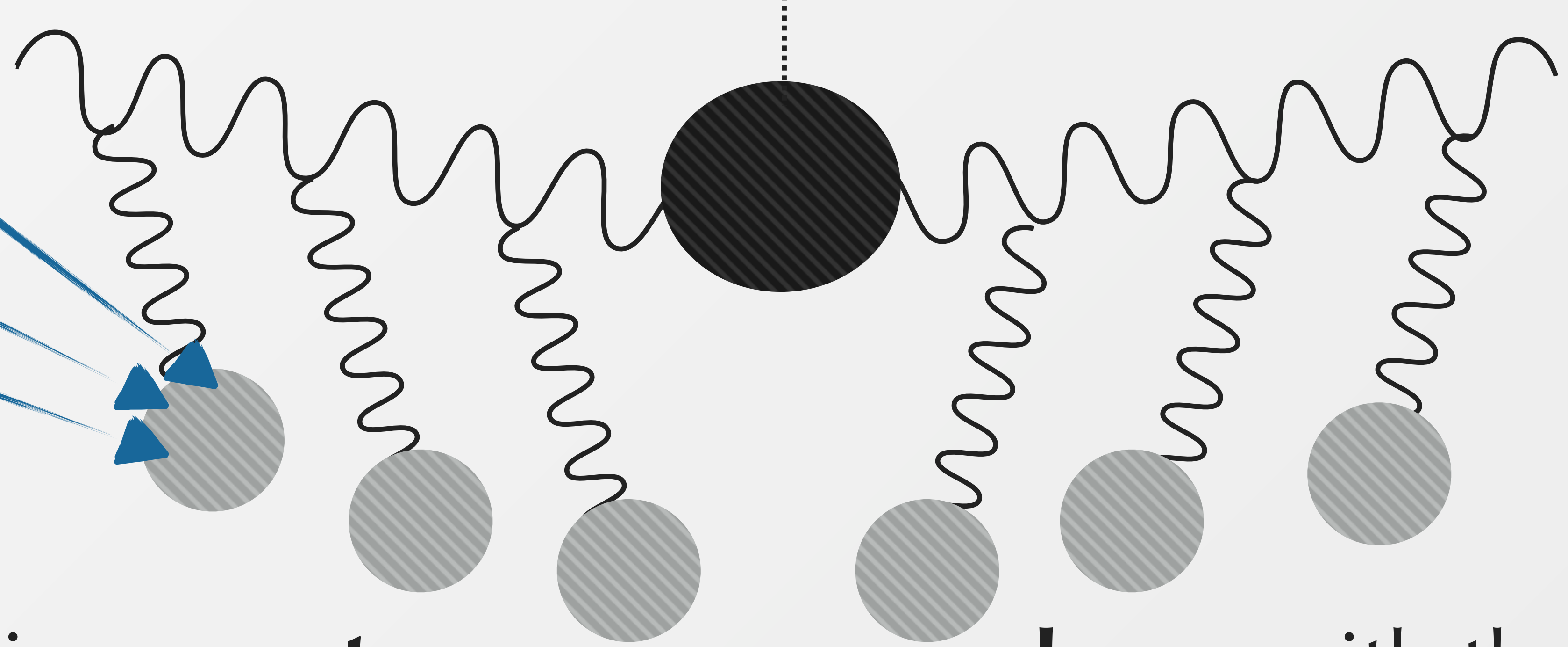
NLL

NNLL

*expression valid for inclusive observables

$$|\tilde{M}(k_1)|^2 = \frac{|M(k_1)|^2}{|M_B|^2} = |M(k_1)|^2$$

$$|\tilde{M}(k_1, k_2)|^2 = \frac{|M(k_1, k_2)|^2}{|M_B|^2} - \frac{1}{2!} |M(k_1)|^2 M|(k_2)|^2$$



Upon integration over the phase space, the expansion can be put in a **one to one correspondence** with the logarithmic structure

Systematic recipe to include terms up to the desired logarithmic accuracy

Resummation in direct space: the p_t case

2. Exploit rIRC safety to single out the IRC singularities of the real matrix element and achieve the **cancellation of the exponentiated divergences** of virtual origin

Introduce a slicing parameter $\epsilon \ll 1$ such that all inclusive blocks with $k_{t,i} < \epsilon k_{t,1}$, with $k_{t,1}$ hardest emission, can be neglected in the computation of the observable

$$\Sigma(v) = \int d\Phi_B |\mathcal{M}_B(\Phi_B)|^2 \mathcal{V}(\Phi_B)$$

unresolved emissions

$$\times \int [dk_1] |\mathcal{M}(k_1)|^2_{\text{inc}} \left(\sum_{l=0}^{\infty} \frac{1}{l!} \int \prod_{j=2}^{l+1} [dk_j] |\mathcal{M}(k_j)|^2_{\text{inc}} \Theta(\epsilon V(k_1) - V(k_j)) \right)$$

$$\times \left(\sum_{m=0}^{\infty} \frac{1}{m!} \int \prod_{i=2}^{m+1} [dk_i] |\mathcal{M}(k_i)|^2_{\text{inc}} \Theta(V(k_i) - \epsilon V(k_1)) \Theta\left(v - V(\Phi_B, k_1, \ldots, k_{m+1})\right) \right)$$

resolved emissions

Unresolved emission doesn't contribute to the evaluation of the observable: it can be exponentiated directly and employed to cancel the virtual divergences, giving rise to a Sudakov radiator

$$\mathcal{V}(\Phi_B) \exp\left\{ \int [dk] |\mathcal{M}(k)|^2_{\text{inc}} \Theta(\epsilon V(k_1) - V(k)) \right\} \simeq e^{-R(\epsilon V(k_1))}$$

Resummation in direct space: the p_t case

Result at NLL accuracy can be written as

$$\Sigma(v) = \sigma^{(0)} \int \frac{dv_1}{v_1} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(\epsilon v_1)} R'\left(v_1\right) \qquad v_i = V(k_i), \quad \zeta_i = v_i / v_1$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'\left(\zeta_i v_1\right) \, \Theta\left(v - V(\Phi_B, k_1, \ldots, k_{n+1})\right)$$

Formula can be evaluated with Monte Carlo method; dependence on ϵ vanishes exactly and result is finite in four dimensions

It contains **subleading effect** which in the original CAESAR approach are disposed of by expanding R and R' around v

$$R(\epsilon v_1) = R(v) + \frac{dR(v)}{d\ln(1/v)} \ln \frac{v}{\epsilon v_1} + \mathcal{O}\left(\ln^2 \frac{v}{\epsilon v_1}\right)$$

$$R'\left(v_i\right) = R'(v) + \mathcal{O}\left(\ln \frac{v}{v_i}\right)$$

Not possible! valid only if the ratio v_i/v remains of order one in the whole emission phase space, but for observables which feature kinematic cancellations there are configurations with $v_i \gg v$. **Subleading effects necessary**

Resummation in direct space: the p_t case

Result at NLL accuracy can be written as

$$\Sigma(v) = \sigma^{(0)} \int \frac{dv_1}{v_1} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(\epsilon v_1)} R'\left(v_1\right) \qquad\qquad v_i = V(k_i), \quad \zeta_i = v_i/v_1$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'\left(\zeta_i v_1\right) \, \Theta\left(v - V(\Phi_B, k_1, \ldots, k_{n+1})\right)$$

Formula can be evaluated with Monte Carlo method; dependence on ϵ vanishes exactly and result is finite in four dimensions

Convenient to perform an expansion around k_{t1} (more efficient and simpler implementation)

$$R(\epsilon k_{t1}) = R(k_{t1}) + \frac{dR(k_{t1})}{d \ln(1/k_{t1})} \ln \frac{1}{\epsilon} + \mathcal{O}\left(\ln^2 \frac{1}{\epsilon}\right)$$

$$R'\left(k_{ti}\right) = R'(k_{t1}) + \mathcal{O}\left(\ln \frac{k_{t1}}{k_{ti}}\right)$$

Subleading effects retained: no divergence at small v, power-like behaviour respected

Logarithmic accuracy defined in terms of $\ln(M/k_{t1})$

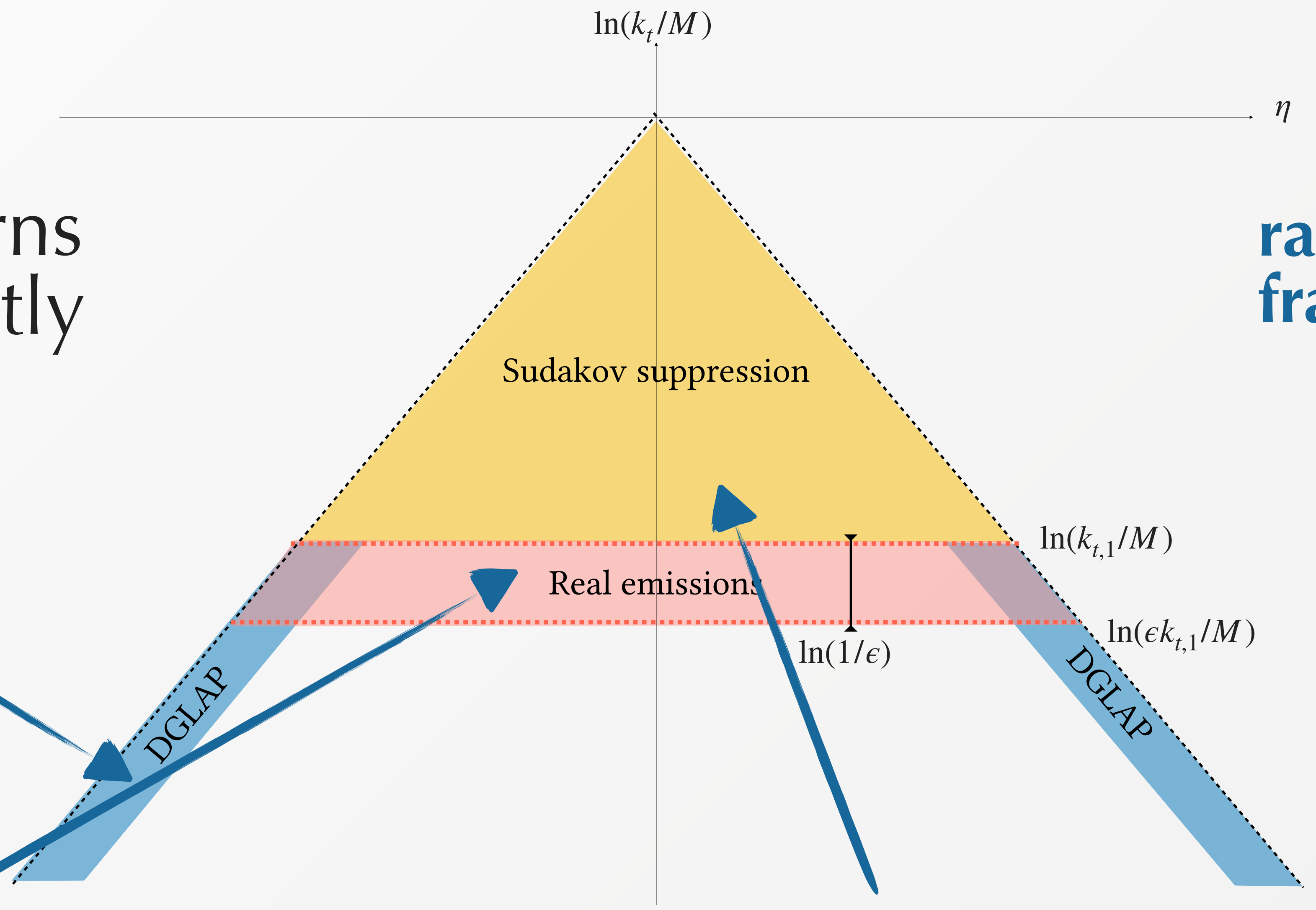
Result formally equivalent to the b-space formulation

Parton luminosities

Consider configurations in which emissions are ordered in $k_{t,i}$, $k_{t,1}$ hardest emission

Phase space for each secondary emission can be depicted in the Lund diagram

DGLAP evolution governs the radiation in the strictly collinear limit



rapidity in the centre-of-mass frame of the incoming partons

resolved emissions live in this strip

remaining **unresolved real emissions** are combined with the virtual corrections to give rise to Sudakov suppression

- DGLAP evolution can be performed **inclusively** up to $\epsilon k_{t,1}$ thanks to rIRC safety
- In the **overlapping region** hard-collinear emissions modify the observable's value: the evolution should be performed exclusively (unintegrated in k_t)
- At NLL the real radiation can be approximated with its soft limit: DGLAP can be performed inclusively up to $k_{t,1}$ (i.e. one can evaluate $\mu_F = k_{t,1}$)

Beyond NLL

Extension to NNLL and beyond requires the systematic inclusion of the correlated blocks necessary to achieve the desired logarithmic accuracy

Moreover, one needs to **relax a series of assumptions** which give rise to subleading corrections neglected at NLL (for instance, exact rapidity bounds). These corrections can be included systematically by including additional terms in the expansion

$$R(\epsilon v_1) = R(v_1) + \frac{dR(v_1)}{d\ln(1/v_1)} \ln\frac{1}{\epsilon} + \mathcal{O}\left(\ln^2\frac{1}{\epsilon}\right)$$

Finally, one needs to specify a complete treatment for **hard-collinear radiation.** Starting at NNLL one or more real emissions can be hard and collinear to the emitting leg, and the available phase space for subsequent real emissions changes

Two classes of contributions:

- one soft by construction and which is analogous to the R′ contribution

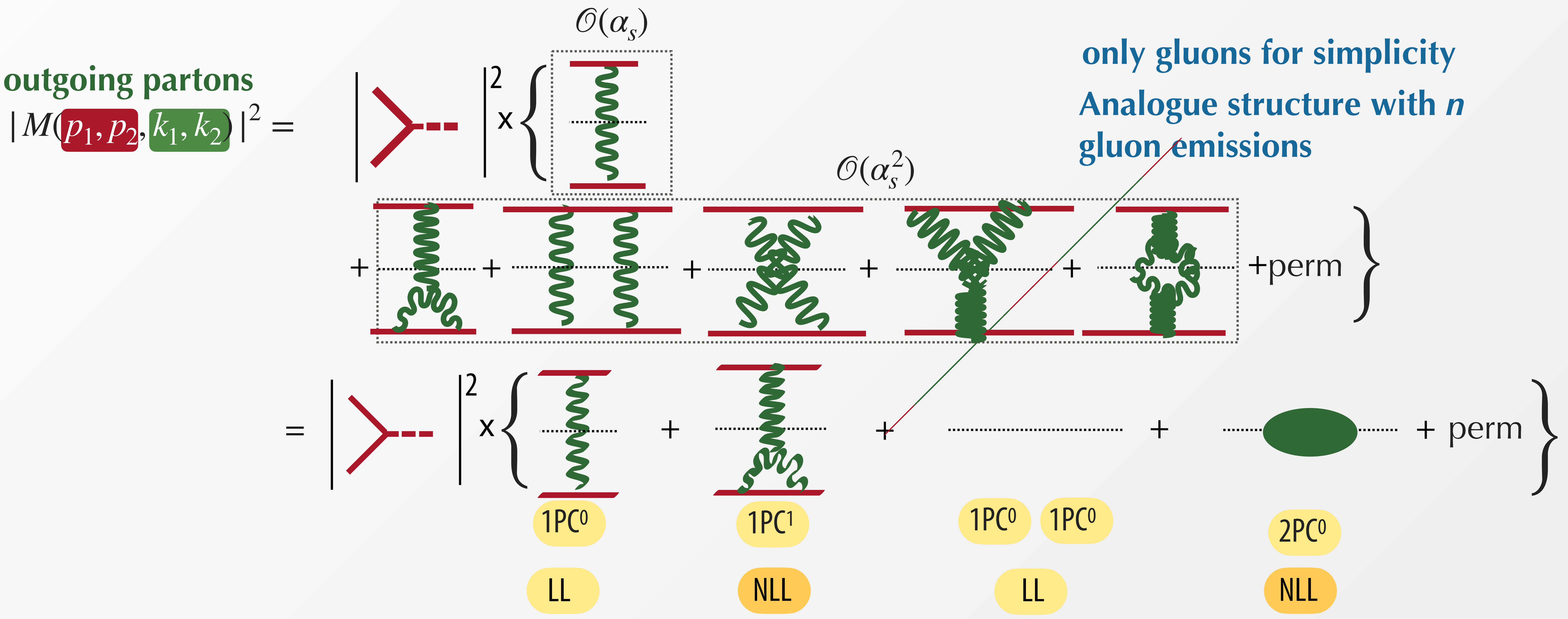
$$R'\left(v_i\right) = R'(v_1) + \mathcal{O}\left(\ln\frac{v_1}{v_i}\right)$$

- another hard and collinear (exclusive DGLAP step): last step of DGLAP evolution must be performed unintegrated in k_t

Logarithmic counting

Necessary to establish a **well defined logarithmic counting**: possibile to do that by decomposing the squared amplitude in terms of n-particle correlated blocks (**nPC**)

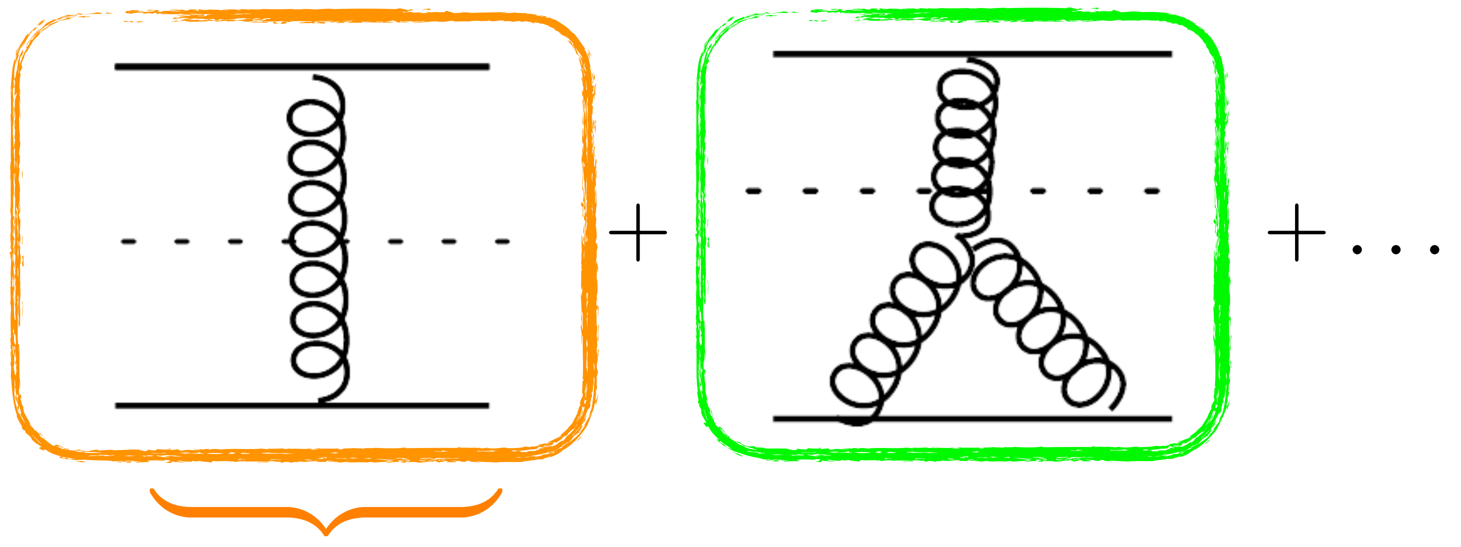
e.g. $pp \to H$ + emission of up to 2 (soft) gluons $O(\alpha_s{}^2)$



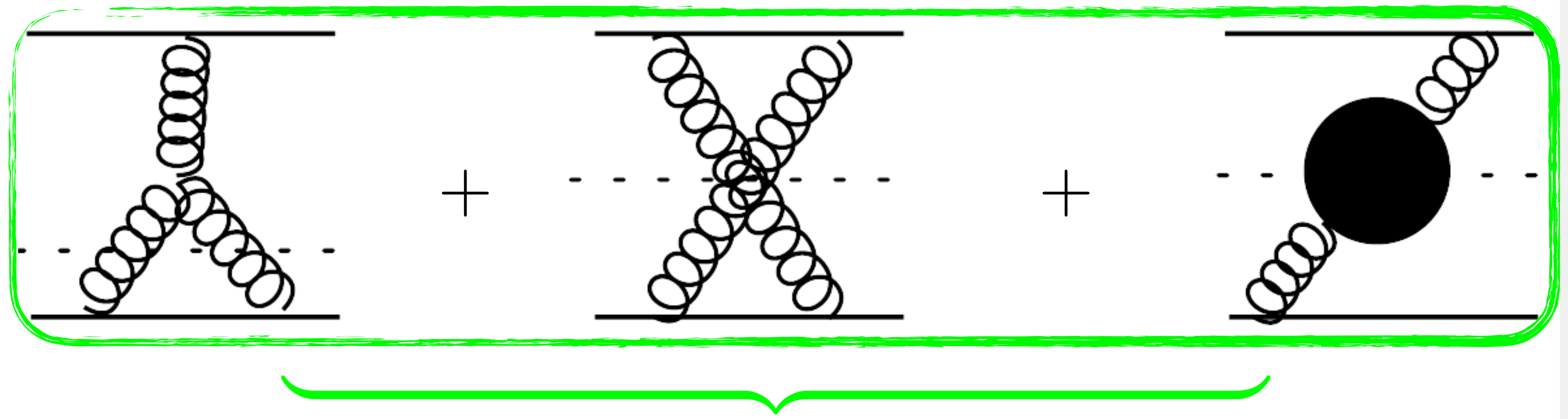
Logarithmic counting defined in terms of **nPC blocks** (owing to rIRC safety of the observable)

Logarithmic counting: correlated blocks

$$|\tilde{M}(k_a)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} = |M(k_a)|^2 \longrightarrow$$



$$|\tilde{M}(k_a, k_b)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} - \underbrace{\frac{1}{2!}|M(k_a)|^2 |M(k_b)|^2}_{\alpha_s^2 L^4} \longrightarrow$$



15 this LL is absorbed in the resummation of $|M(k)|^2$

Thanks to P. Monni

Resummation at NLL accuracy

Final result at NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t,1}}{k_{t,1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} e^{-R(k_{t,1})} \epsilon^{R'(k_{t,1})} \mathscr{L}_{\mathrm{NLL}}(k_{t,1}) R'\left(k_{t,1}\right)$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'\left(\zeta_i k_{t,1}\right) \Theta\left(v - V(\Phi_B, k_1, \ldots, k_{n+1})\right)$$

This formula can be evaluated by means of fast Monte Carlo methods `RadISH (Radiation off Initial State Hadrons)`

Parton luminosity at NLL reads

$$\mathscr{L}_{\mathrm{NLL}}(k_{t,1}) = \sum_c \frac{d|M_B|^2_{c\bar{c}}}{d\Phi_B} f_c\left(x_1, k_{t,1}^2\right) f_{\bar{c}}\left(x_2, k_{t,1}^2\right)$$

At higher logarithmic accuracy, it includes **coefficient functions** and **hard-virtual** corrections

Result at N^{3}LL accuracy

$$
\begin{aligned}
\frac{d\Sigma(v)}{d\Phi_B} &= \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1})\right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \\
&+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1})\right) \\
&\times \left(R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s}\right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s}\right) \\
&+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\} \\
&+ \frac{1}{2} \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}} \frac{d\phi_{s1}}{2\pi} \int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}} \frac{d\phi_{s2}}{2\pi} R'(k_{t1}) \\
&\times \Bigg\{ \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \left(R''(k_{t1})\right)^2 \ln \frac{1}{\zeta_{s1}} \ln \frac{1}{\zeta_{s2}} - \partial_L \mathcal{L}_{\mathrm{NLL}}(k_{t1}) R''(k_{t1}) \left(\ln \frac{1}{\zeta_{s1}} + \ln \frac{1}{\zeta_{s2}}\right) \\
&+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \\
&\times \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1}, k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1})\right) - \\
&\Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\} + \mathcal{O}\left(\alpha_s^n \ln^{2n-6} \frac{1}{v}\right), \qquad (3.18)
\end{aligned}
$$

[Bizon, Monni, Re, LR, Torrielli '17]

All ingredients to perform resummation at **N^{3}LL accuracy** are now available

[Catani *et al.* '11, '12][Gehrmann *et al.* '14][Li, Zhu '16, Vladimirov '16][Moch *et al.* '18, Lee *et al.* '19]

Fixed-order predictions now available at **NNLO**

[A. Gehrmann-De Ridder *et al.* '15, 16, '17][Boughezal *et al.* '15, 16]

Matching with fixed order

Multiplicative matching performed at the **double-cumulant level**

fixed-order double-cumulative result at NNLO

$$\sigma_{\text{NNLO}}(p_\perp^H < p_\perp^{H,\text{v}}, p_\perp^{\text{J}} < p_\perp^{\text{J,v}}) = \sigma_{\text{NNLO}} - \int \Theta(p_\perp^H > p_\perp^{H,\text{v}}) \vee \Theta(p_\perp^{\text{J}} > p_\perp^{\text{J,v}}) d\sigma_{H+J,\text{NLO}}$$

double-cumulative result at NNLL

$$\sigma_{\text{match}}(p_\perp^H < p_\perp^{H,\text{v}}, p_\perp^{\text{J}} < p_\perp^{\text{J,v}}) = \frac{\sigma_{\text{NNLL}}(p_\perp^H < p_\perp^{H,\text{v}}, p_\perp^{\text{J}} < p_\perp^{\text{J,v}})}{\sigma_{\text{NNLL}}(\{p_\perp^{\text{J,v}}, p_\perp^{H,\text{v}}\} \to \infty)} \left[\sigma_{\text{NNLL}}(\{p_\perp^{\text{J,v}}, p_\perp^{H,\text{v}}\} \to \infty) \frac{\sigma_{\text{NNLO}}(p_\perp^H < p_\perp^{H,\text{v}}, p_\perp^{\text{J}} < p_\perp^{\text{J,v}})}{\sigma_{\text{NNLL,exp}}(p_\perp^H < p_\perp^{H,\text{v}}, p_\perp^{\text{J}} < p_\perp^{\text{J,v}})} \right]_{\mathcal{O}(\alpha_s^2)}$$

asymptotic limit of the NNLL result

expansion of the double-cumulative result at NNLL

- NNLL+NNLO result for $p_\perp^{\text{J,v}}$ recovered for $p_\perp^{\text{H,v}} \to \infty$
- **NNLO constant** included through multiplicative matching (NNLL′ accuracy)

Matching to fixed order: multiplicative matching

Cumulative cross section should reduce to the fixed order at large v

$$\Sigma^{\text{mult}}_{\text{matched}}(v) \sim \Sigma_{\text{res}}(v)\left[\frac{\Sigma_{\text{f.o.}}(v)}{\Sigma_{\text{res}}(v)}\right]_{\text{expanded}}$$

$$\Sigma_{\text{f.o}}(v) = \sigma_{f.o.} - \int_v^\infty \frac{d\sigma}{dv}dv$$

- allows to include constant terms from NNLO (if N^{3}LO total xs available)
- physical suppression at small v cures potential instabilities

To ensure that resummation does not affect the hard region of the spectrum when the matching is performed we introduce **modified logarithms**

This corresponds to restrict the rapidity phase space at large k_t

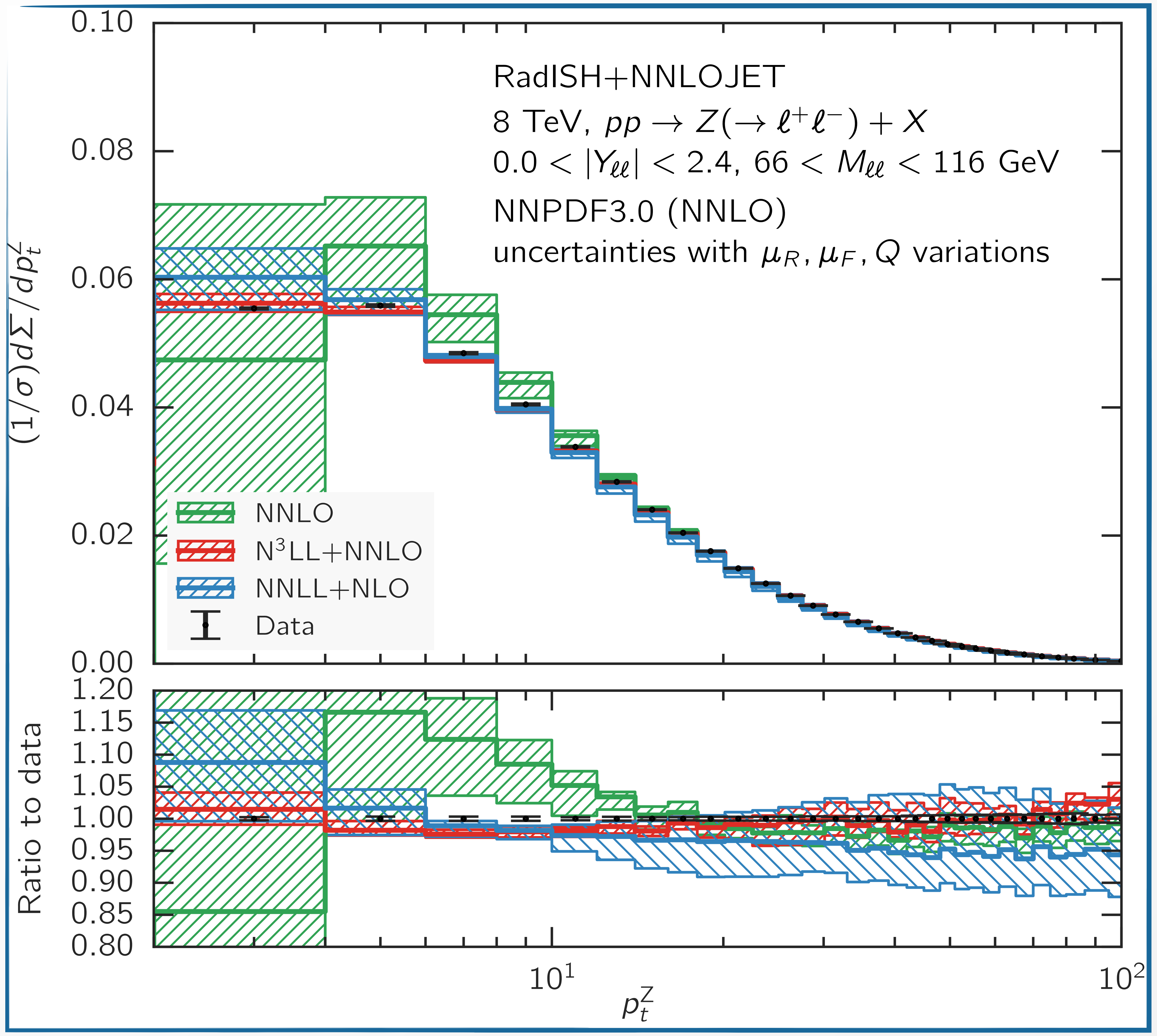
$$\int_{-\ln Q/k_{t,i}}^{\ln Q/k_{t,i}} d\eta \to \int_{-\ln Q/k_{t,1}}^{\ln Q/k_{t,1}} d\eta \to \int_{-\epsilon}^{\epsilon} d\eta \to 0$$

$$\ln(Q/k_{t1}) \to \frac{1}{p}\ln\left(1+\left(\frac{Q}{k_{t1}}\right)^p\right)$$

Q : **perturbative resummation scale** used to probe the size of subleading logarithmic corrections

p : arbitrary matching parameter

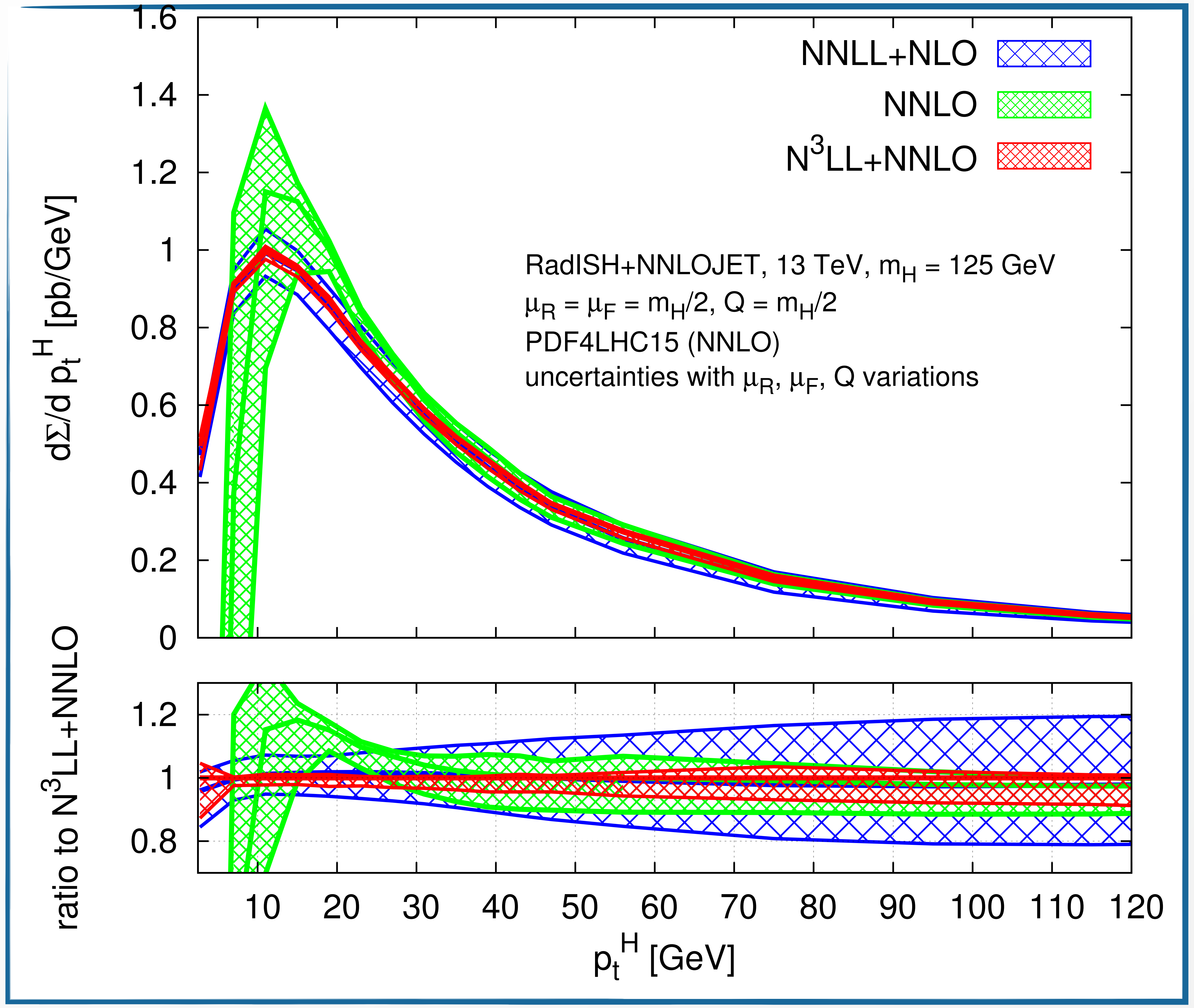
Predictions for the Z spectrum at 8 TeV

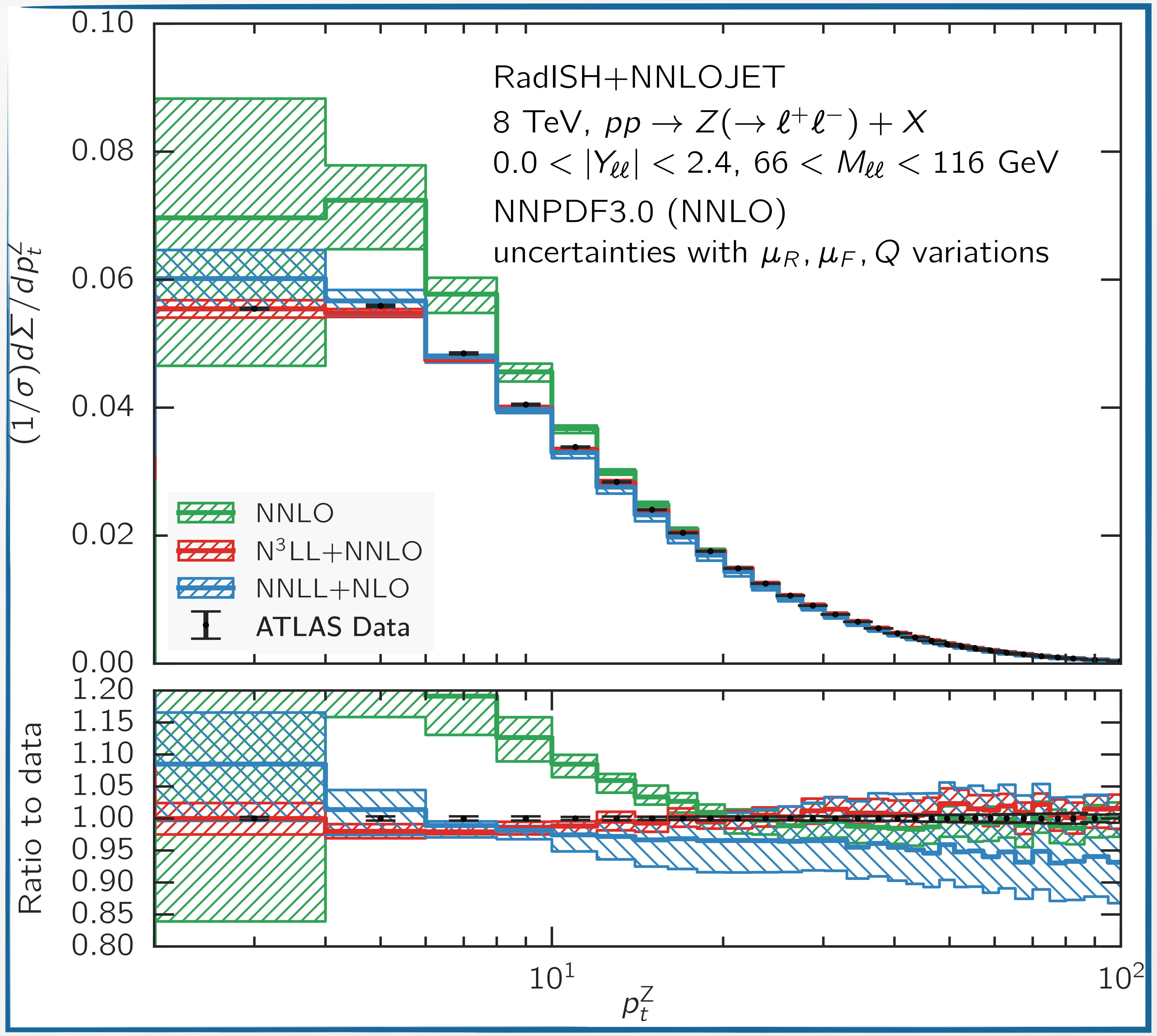


- Good description of the data in all fiducial regions
- Perturbative uncertainty at the few percent level, still does not match the precision of the ATLAS data

Resummation of the transverse momentum spectrum at N^{3}LL+NNLO

N^{3}LL result matched to NNLO H+j, Z+j, W$^{\pm}$+j **[Bizon, LR *et al.* '18, '19]**





Theoretical predictions for Z and W observables at 13 TeV

Bizon, Gehrmann-De Ridder, Gehrmann, Glover, Huss, Monni, Re, LR, Walker, 190x.xxxx

Results obtained using the following fiducial cuts (agreed with ATLAS)

$$p_t^{\ell^\pm} > 25\,\mathrm{GeV}, \quad |\eta^{\ell^\pm}| < 2.5, \quad 66\,\mathrm{GeV} < M_{\ell\ell} < 116\,\mathrm{GeV}$$

$$p_t^{\ell} > 25\,\mathrm{GeV}, \quad |\eta^{\ell}| < 2.5, \quad \not{E}_T^{\nu_\ell} > 25\,\mathrm{GeV}, \quad m_T > 50\,\mathrm{GeV}$$

using NNPDF3.1 with $\alpha_s(M_Z)$=0.118 and setting the central scales to

$$\mu_R = \mu_F = M_T = \sqrt{M_{\ell\ell'}^2 + p_T^2}, \quad Q = \frac{M_{\ell\ell'}}{2}$$

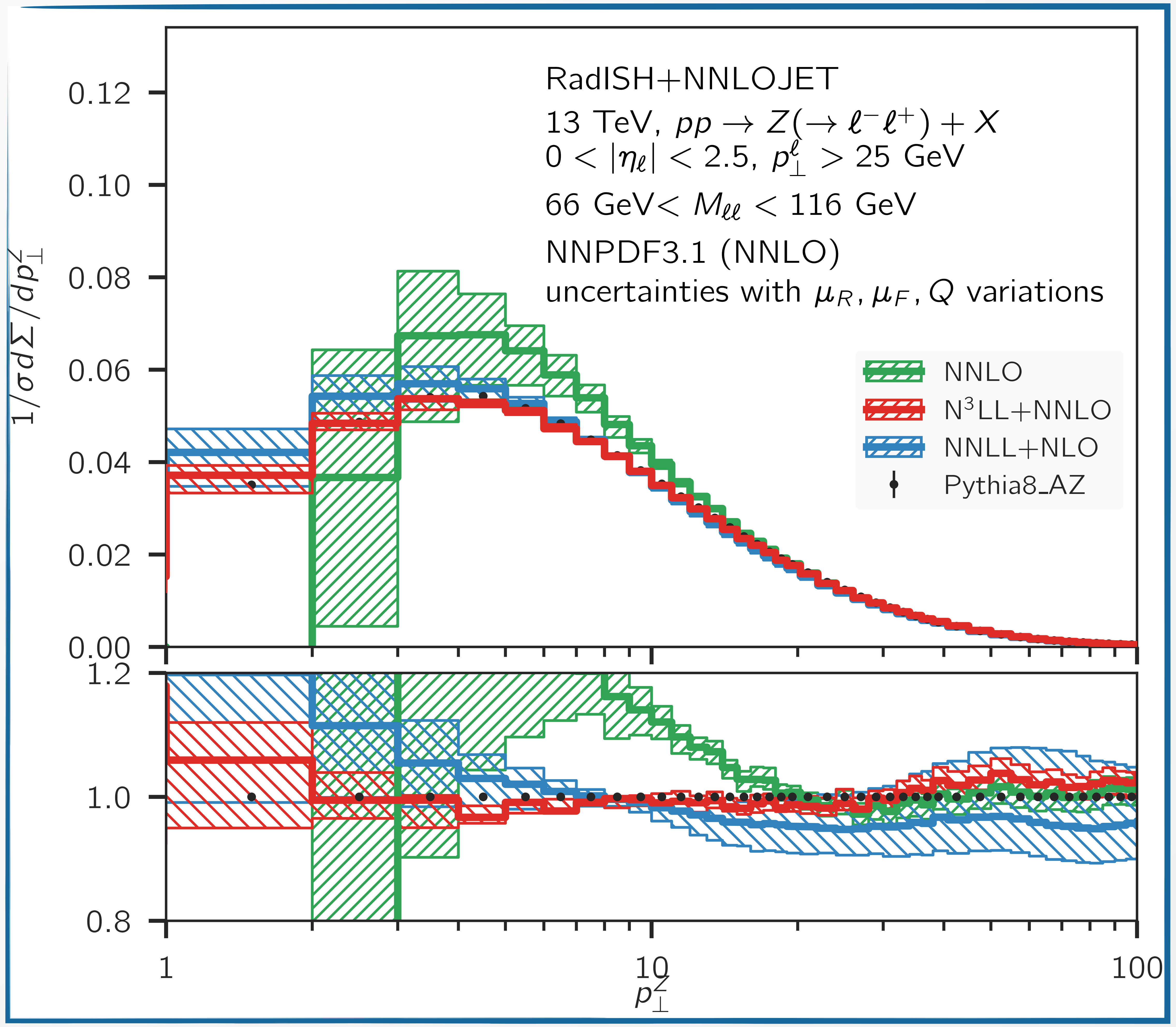
5 flavour (massless) scheme: no HQ effects, LHAPDF PDF thresholds

Scale uncertainties estimated by varying **renormalization** and **factorization** scale by a factor of two around their central value (**7 point variation**) and varying the **resummation** scale by a factor of 2 around its central value for factorization and renormalization scales set to their central value: **9 point envelope**

Matching parameter p set to 4 as a default

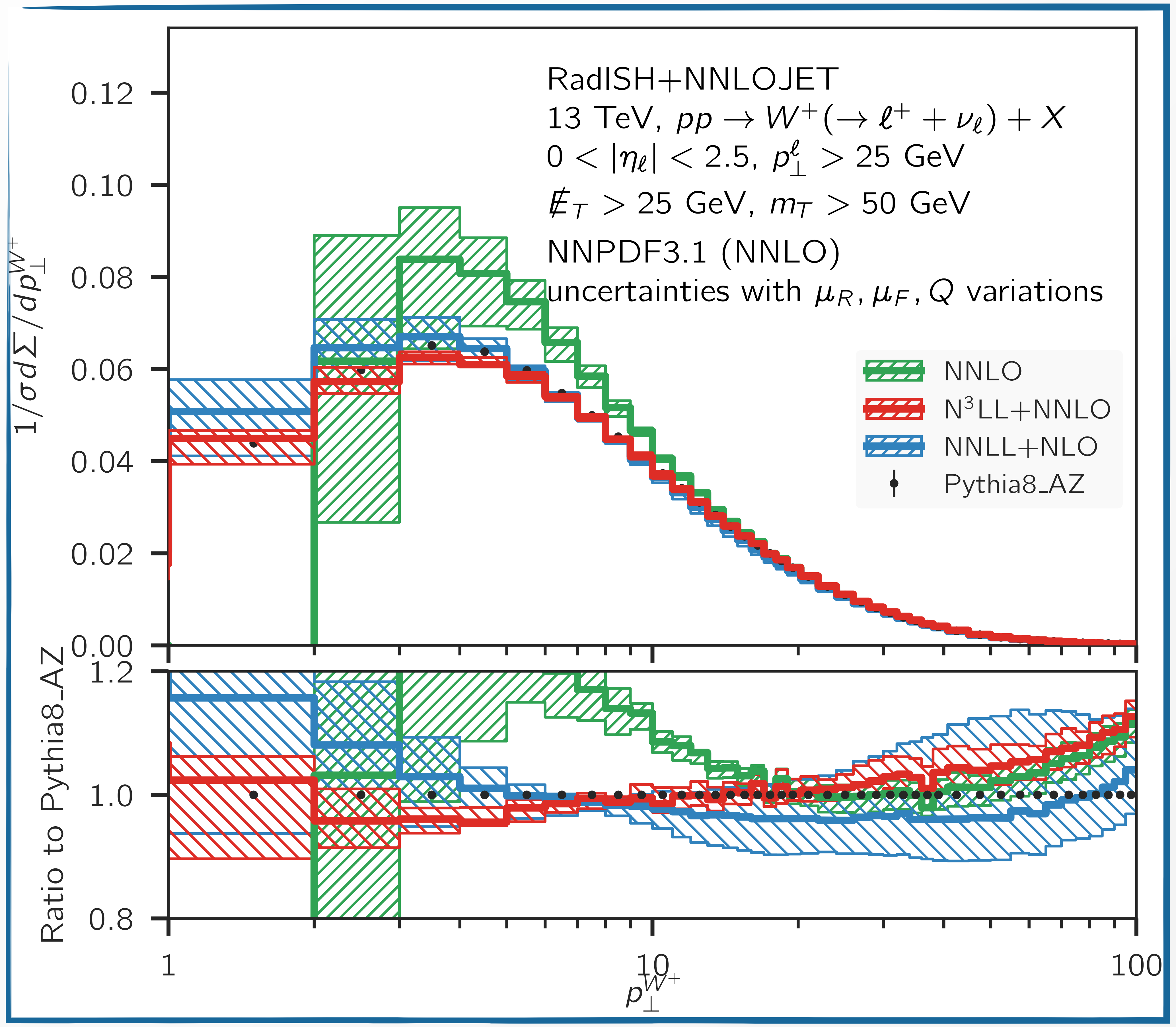
No non perturbative parameters included in the following

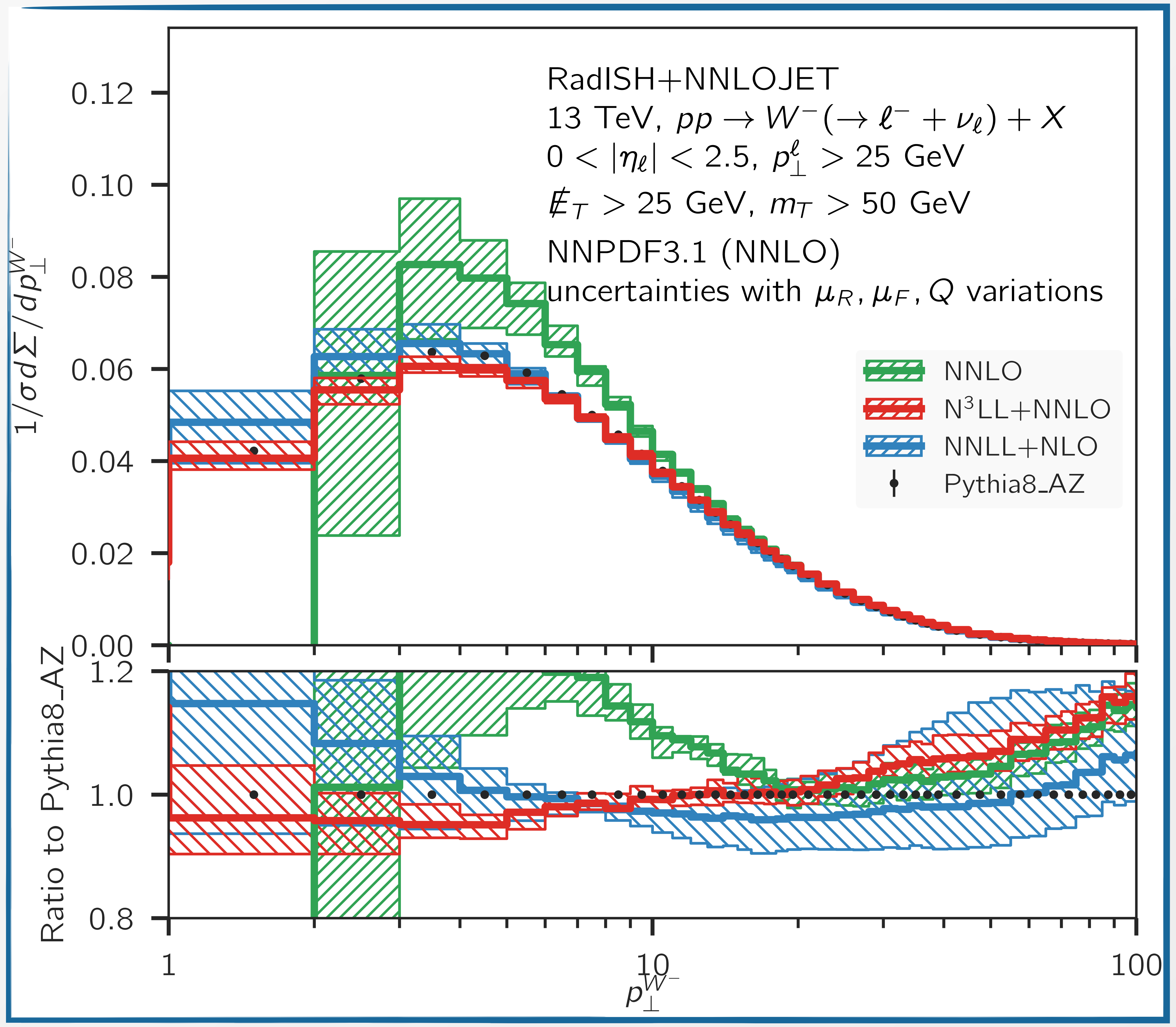
Predictions for the Z spectrum



Thanks to Jan Kretzschmar for providing the PYTHIA8 AZ tune results

Predictions for the *W*+ and *W*- spectra





Ratio of differential distributions

Z and W production share a similar pattern of QCD radiative corrections

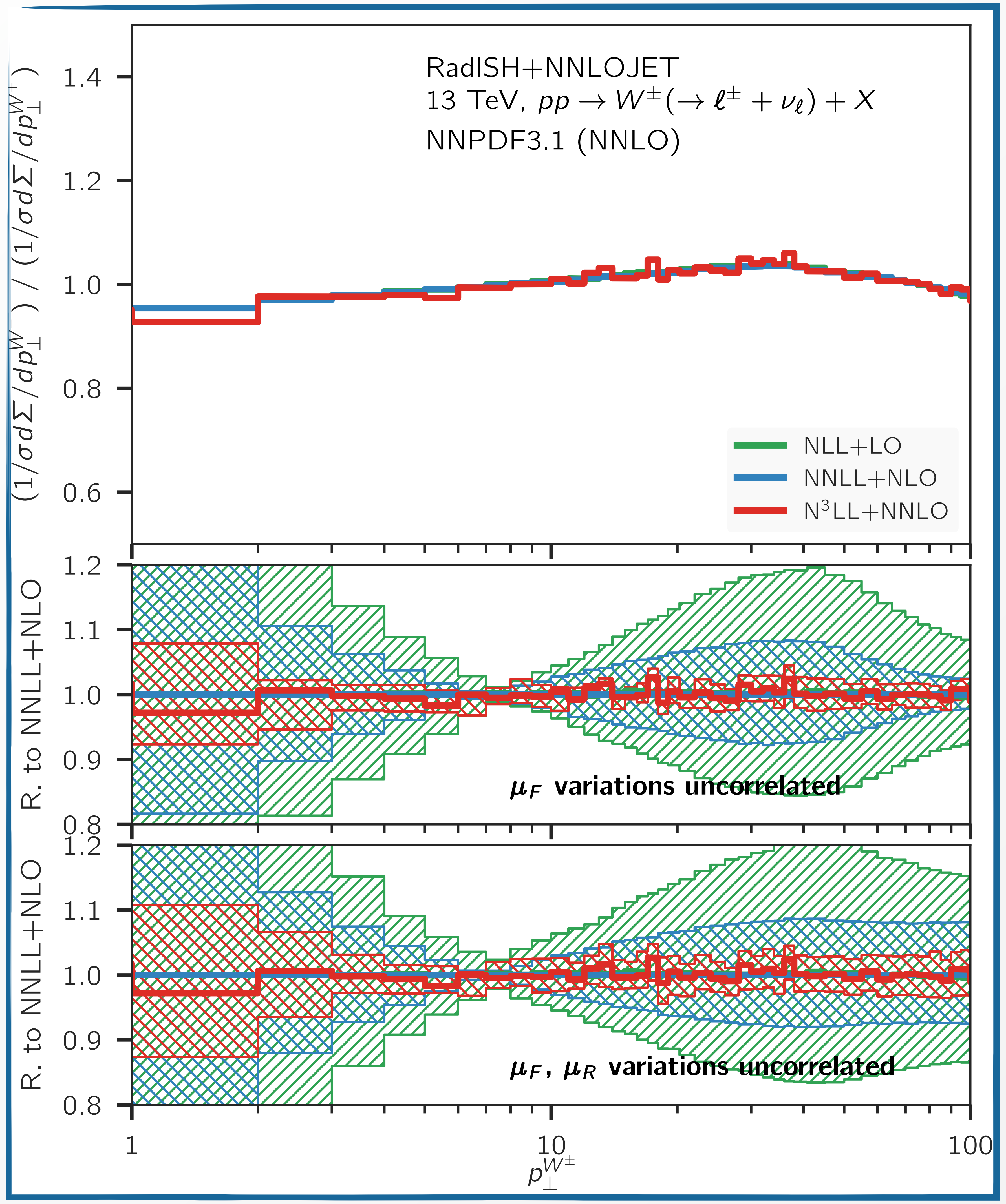
Crucial to understand correlation between Z and W spectra to exploit data-driven predictions

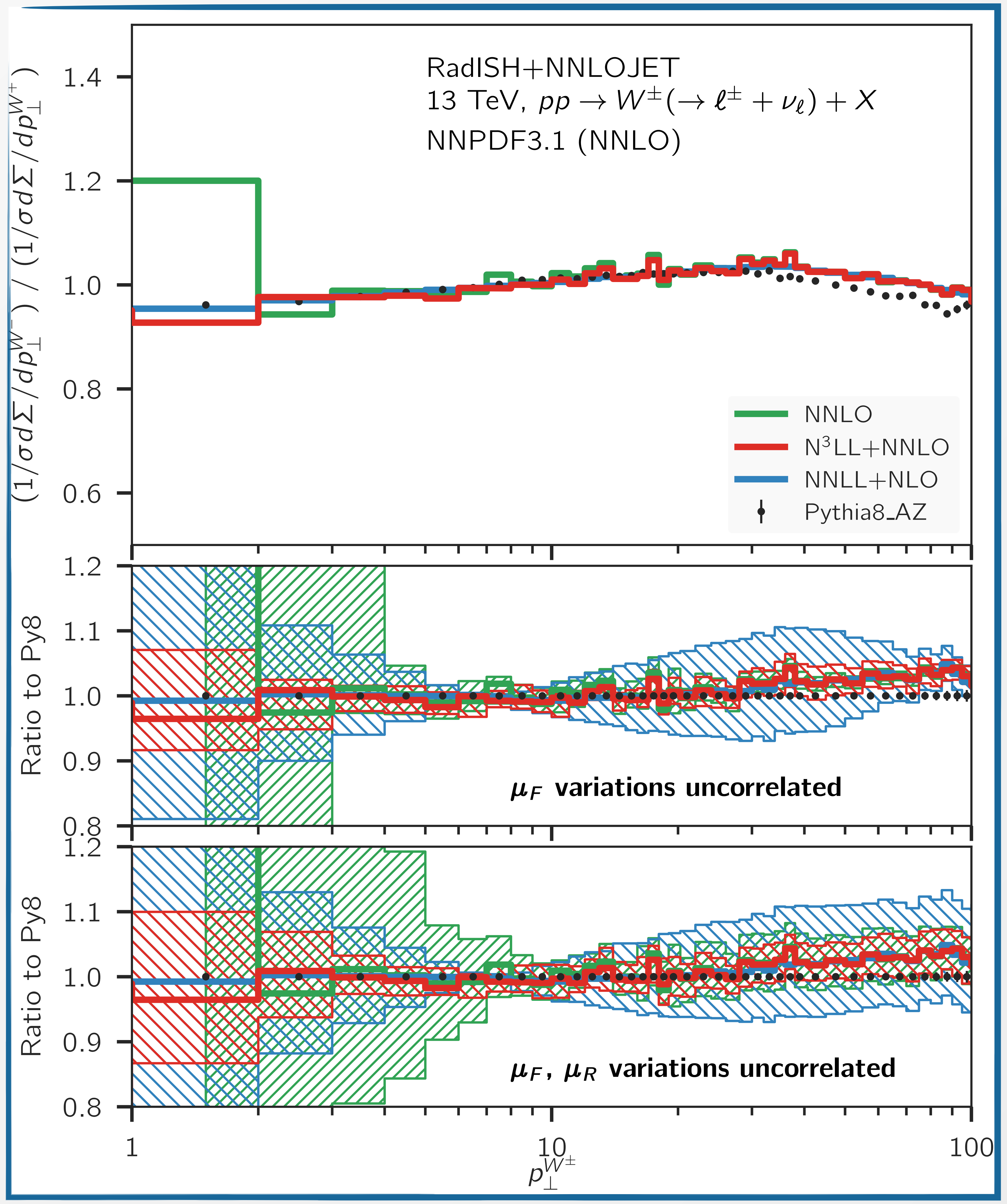
$$\frac{1}{\sigma^W}\frac{d\sigma^W}{p_\perp^W} \sim \frac{1}{\sigma^Z_{\rm data}}\frac{d\sigma^Z_{\rm data}}{p_\perp^Z}\,\frac{\frac{1}{\sigma^W_{\rm theory}}\frac{d\sigma^W_{\rm theory}}{p_\perp^W}}{\frac{1}{\sigma^Z_{\rm theory}}\frac{d\sigma^Z_{\rm theory}}{p_\perp^Z}}$$

Several choices are possible:

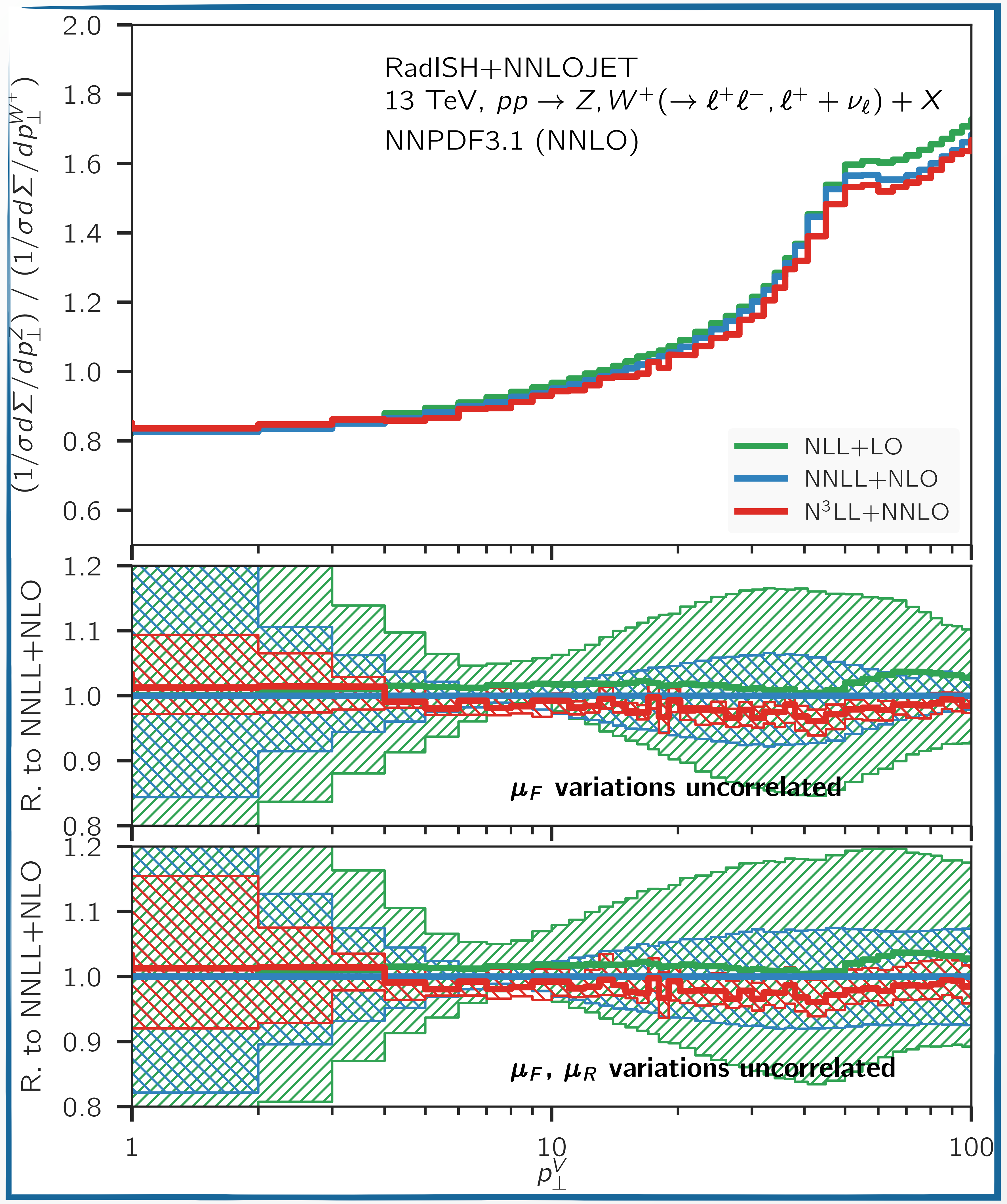
- **Correlate resummation and renormalisation** scale variations, keep **factorisation scale uncorrelated**, while keeping
$$\frac{1}{2} \leq \frac{\mu_{\rm F}^{\rm num}}{\mu_{\rm F}^{\rm den}} \leq 2$$
- More **conservative** estimate: vary both **renormalisation and factorisation scales in an uncorrelated** way with
$$\frac{1}{2} \leq \frac{\mu^{\rm num}}{\mu^{\rm den}} \leq 2$$

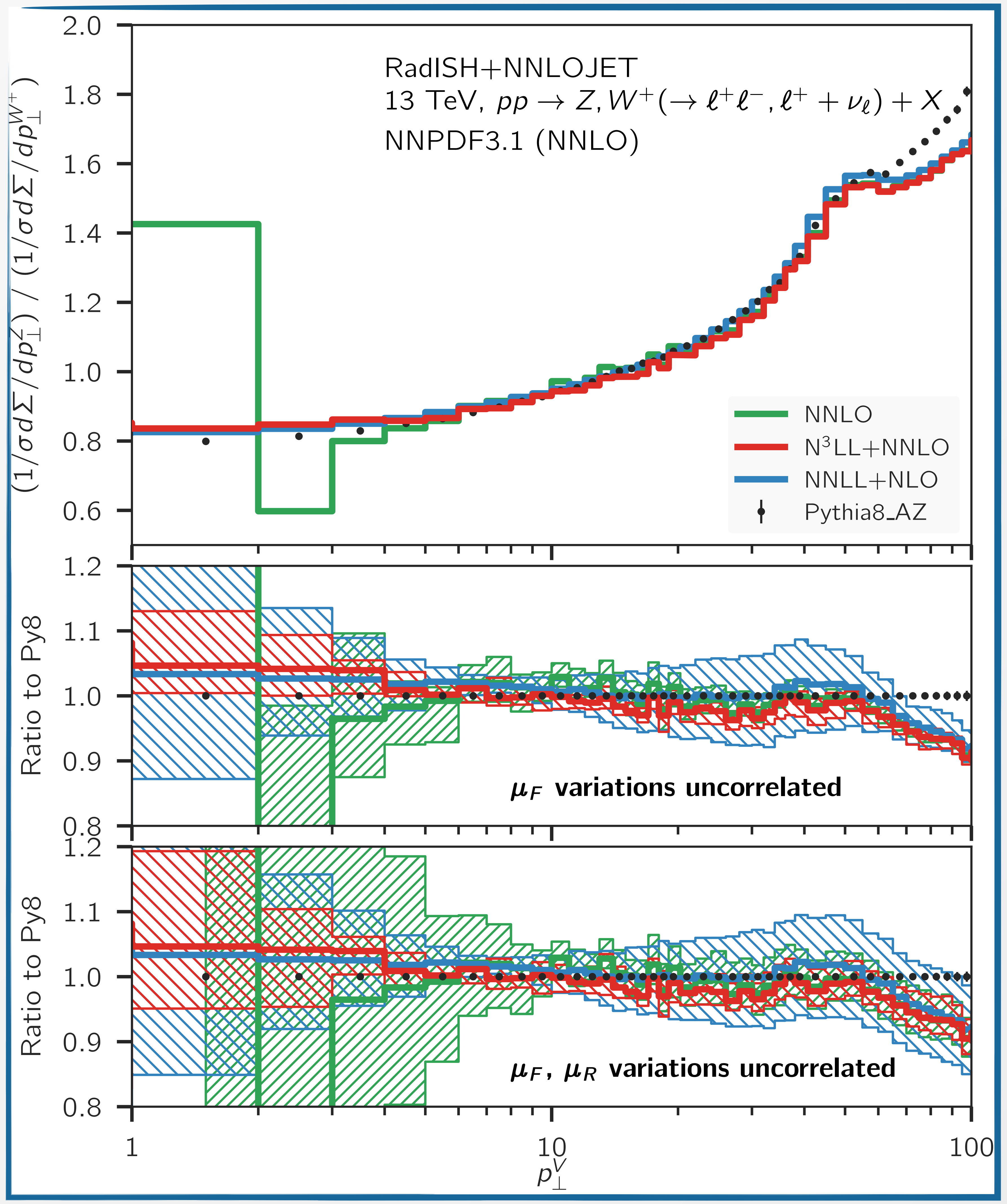
Results for W^-/W^+ ratio





Results for Z/W^+ ratio





Equivalence with *b*-space formulation

$$\frac{d\Sigma(v)}{d\Phi_B} = \int_{\mathscr{C}_1}\frac{dN_1}{2\pi i}\int_{\mathscr{C}_2}\frac{dN_2}{2\pi i}x_1^{-N_1}x_2^{-N_2}\sum_{c_1,c_2}\frac{d|M_B|^2_{c_1c_2}}{d\Phi_B}\mathbf{f}^T_{N_1}(\mu_0)\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)\mathbf{f}_{N_2}(\mu_0)$$

unresolved emission + virtual corrections

$$\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right]\int_0^M\frac{dk_{t1}}{k_{t1}}\int_0^{2\pi}\frac{d\phi_1}{2\pi}$$

$$\times\, e^{-\mathbf{R}(\epsilon k_{t1})}\exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$

Result valid for all inclusive observables (e.g. p_t, φ^*)

resolved emission

$$\sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$

$$\times\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=2}^{n+1}\int_\epsilon^1\frac{d\zeta_i}{\zeta_i}\int_0^{2\pi}\frac{d\phi_i}{2\pi}\sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi}\mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right)$$

$$\times\,\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)$$

Formulation **equivalent to *b*-space** result (up to a **scheme change** in the anomalous dimensions)

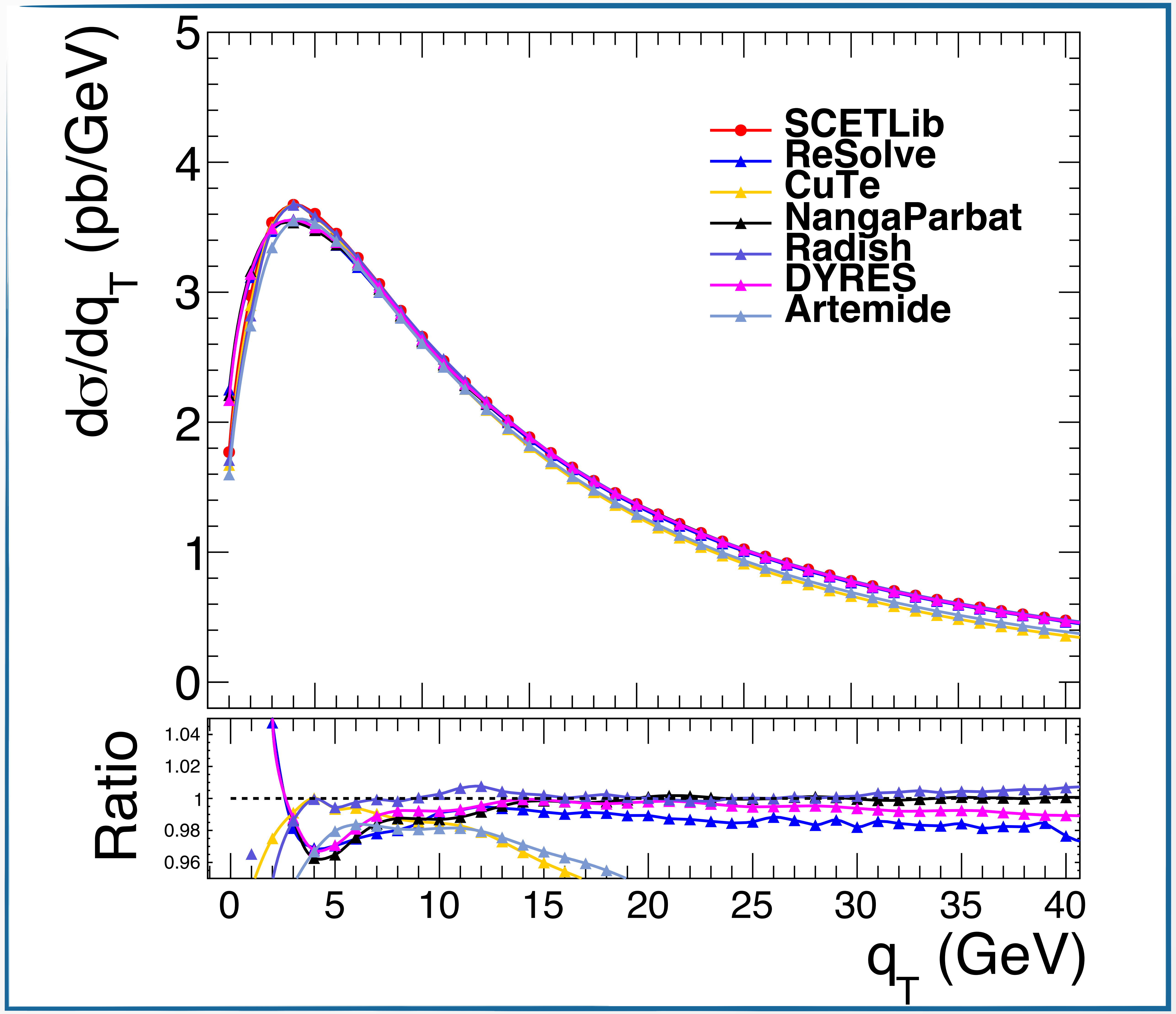
$$\frac{d^2\Sigma(v)}{d\Phi_B dp_t} = \sum_{c_1,c_2}\frac{d|M_B|^2_{c_1c_2}}{d\Phi_B}\int b\,db\,p_tJ_0(p_tb)\,\mathbf{f}^T(b_0/b)\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(b_0/b))H(M)\mathbf{C}^{c_2}_{N_2}(\alpha_s(b_0/b))\mathbf{f}(b_0/b)$$

$$\times\exp\left\{-\sum_{\ell=1}^{2}\int_0^M\frac{dk_t}{k_t}\mathbf{R}'_\ell(k_t)(1-J_0(bk_t))\right\}$$

$$(1-J_0(bk_t)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12}\frac{\partial^3}{\partial\ln(Mb/b_0)^3}\Theta(k_t - \frac{b_0}{b})$$

N^3LL effect: absorbed in the definition of H_2, B_3, A_4 coefficients wrt to CSS

Equivalence with *b*-space formulation

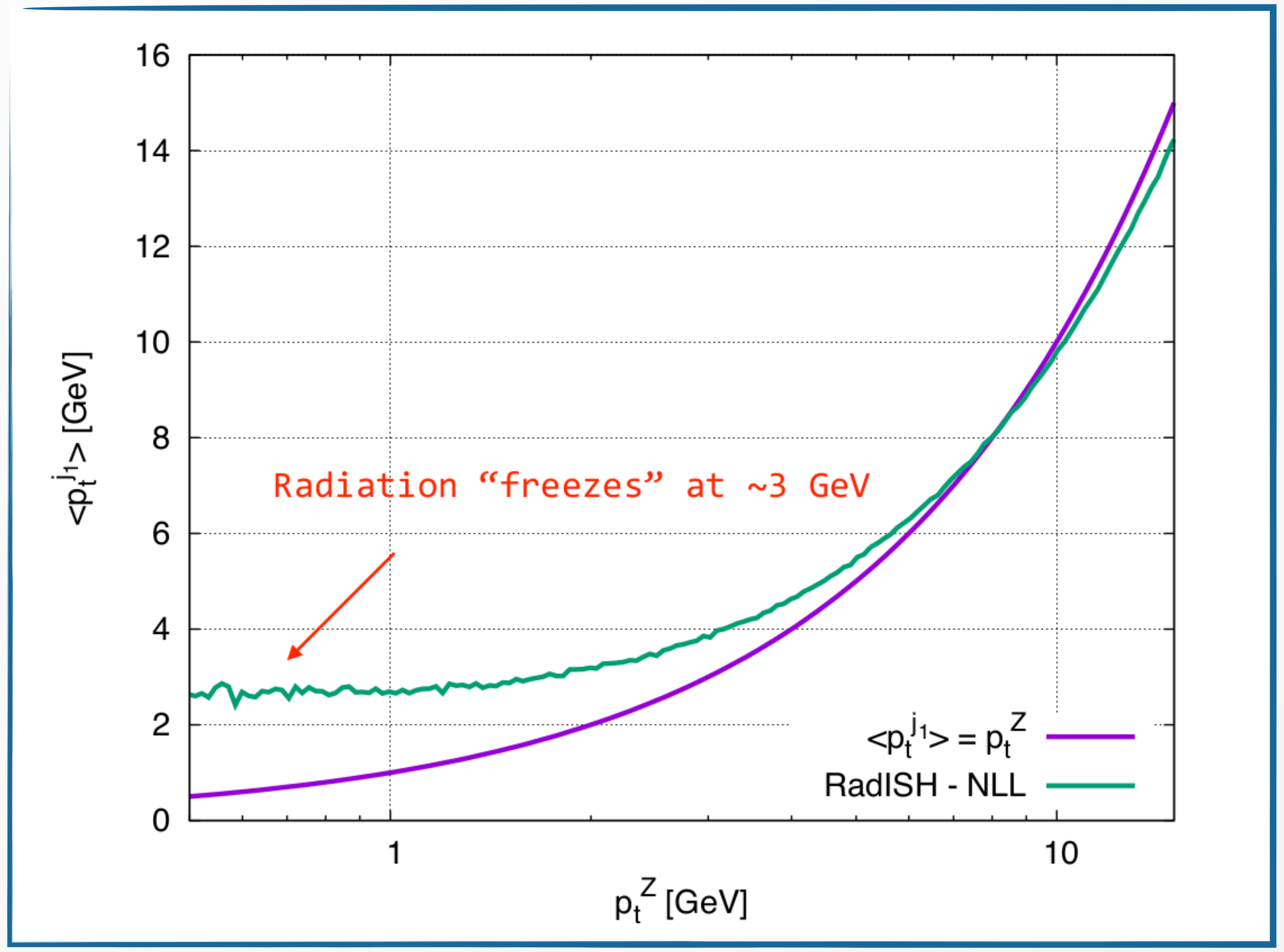


The Landau pole and the small p_T limit

Running coupling $\alpha_s(k_{t,1}{}^2)$ and Sudakov radiator hit Landau pole at

$$\alpha_s(\mu_R^2)\beta_0 \ln Q/k_{t1} = \frac{1}{2} \qquad k_{t1} \sim 0.01\,\text{GeV}, \quad \mu_R = Q = m_Z$$

Only real cutoff in the calculation: emission probability is set to zero below this scale and parton densities are frozen.



At small p_t the large azimuthal cancellations dominate over the Sudakov suppression: the cutoff is never an issue in practice

$$\frac{d^2\Sigma(v)}{dp_t d\Phi_B} \simeq 2\sigma^{(0)}(\Phi_B)p_t\left(\frac{\Lambda_{QCD}^2}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}}$$

Thanks to P. Monni

Behaviour at small p_t

Explicit evaluation shows that the Parisi-Petronzio perturbative scaling at small p_t is reproduced. At NLL, Drell-Yan pair production, n_f=4

$$\frac{d^2\Sigma(v)}{dp_t d\Phi_B} = 4\,\sigma^{(0)}(\Phi_B)\,p_t \int_{\Lambda_{\rm QCD}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2\sigma^{(0)}(\Phi_B) p_t \left(\frac{\Lambda_{\rm QCD}^2}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}}$$

As now higher logarithmic terms (up to N^3LL) are under control, the coefficient of this scaling can be systematically improved in *perturbation* theory (non-perturbative effects – of the same order – not considered)

N^3LL calculation allows one to have control over the terms of relative order $O(\alpha_s^2)$. Scaling $L \sim 1/\alpha_s$ valid in the deep infrared regime.

Numerical implementation

$$\frac{d\Sigma(p_t)}{d\Phi_B} = \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R'(k_{t1})} \mathcal{L}_{\rm NLL}(k_{t1})\right) \times$$
$$\times \underbrace{\epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1}) \right) \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|)}_{\equiv \int d\mathcal{Z}[\{R', k_i\}] \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|)} .$$

- $L = \ln(M/k_{t1})$; luminosity $\mathcal{L}_{\rm NLL}(k_{t1}) = \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} f_{c_1}(x_1, k_{t1}) f_{c_2}(x_2, k_{t1})$.
- $\int d\mathcal{Z}[\{R', k_i\}]\Theta$ finite as $\epsilon \to 0$:

$$\epsilon^{R'(k_{t1})} = 1 - R'(k_{t1}) \ln(1/\epsilon) + \ldots = 1 - \int_{\epsilon k_{t1}}^{k_{t1}} R'(k_{t1}) + \ldots,$$

$$\int d\mathcal{Z}[\{R', k_i\}]\Theta = \left[1 - \int_{\epsilon k_{t1}}^{k_{t1}} R'(k_{t1}) + \ldots\right] \left[\Theta(p_t - |\vec{k}_{t1}|) + \int_{\epsilon k_{t1}}^{k_{t1}} R'(k_{t1}) \Theta(p_t - |\vec{k}_{t1} + \vec{k}_{t2}|) + \ldots\right]$$
$$= \Theta(p_t - |\vec{k}_{t1}|) + \underbrace{\int_0^{k_{t1}}}_{\epsilon \to 0} R'(k_{t1}) \underbrace{\left[\Theta(p_t - |\vec{k}_{t1} + \vec{k}_{t2}|) - \Theta(p_t - |\vec{k}_{t1}|)\right]}_{\text{finite: real-virtual cancellation}} + \ldots$$

- Evaluated with Monte Carlo techniques: $\int d\mathcal{Z}[\{R', k_i\}]$ is generated as a parton shower over secondary emissions.

Thanks to P. Torrielli

Numerical implementation

- Secondary radiation:

$$d\mathcal{Z}[\{R', k_i\}] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) \right) \epsilon^{R'(k_{t1})}$$

$$= \sum_{n=0}^{\infty} \left(\prod_{i=2}^{n+1} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int_{\epsilon k_{t1}}^{k_{t(i-1)}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) \right) \epsilon^{R'(k_{t1})},$$

$$\epsilon^{R'(k_{t1})} = e^{-R'(k_{t1}) \ln 1/\epsilon} = \prod_{i=2}^{n+2} e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}},$$

with $k_{t(n+2)} = \epsilon k_{t1}$.

- Each secondary emissions has differential probability

$$dw_i = \frac{d\phi_i}{2\pi} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} = \frac{d\phi_i}{2\pi} d\left(e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} \right).$$

- $k_{t(i-1)} \geq k_{ti}$. Scale k_{ti} extracted by solving $e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} = r$, with r random number extracted uniformly in $[0, 1]$. Shower ordered in k_{ti}.

- Extract ϕ_i randomly in $[0, 2\pi]$.

Thanks to P. Torrielli

Joint resummation in direct space

$$\sigma_{\rm incl}^{\rm NNLL}(p_t^{\rm J,v}, p_t^{\rm H,v}) = \int_0^{p_t^{\rm J,v}} \frac{dk_{t,1}}{k_{t,1}} \frac{d\phi_1}{2\pi} \int d\mathcal{Z} \Bigg\{ \frac{d}{d\,L_{t,1}} \left[-e^{-R_{\rm NNLL}(L_{t,1})} \mathcal{L}_{\rm NNLL}\left(\mu_{\rm F} e^{-L_{t,1}}\right)\right] \Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1}|\Big)$$
$$+ e^{-R_{\rm NLL}(L_{t,1})} \hat{R}'(k_{t,1}) \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{d\phi_{s_1}}{2\pi} \left[\left(\delta \hat{R}'(k_{t,1}) + \hat{R}''(k_{t,1}) \ln \frac{k_{t,1}}{k_{t,s_1}} \right) \mathcal{L}_{\rm NLL}\left(\mu_{\rm F} e^{-L_{t,1}}\right) - \frac{d}{d\,L_{t,1}} \mathcal{L}_{\rm NLL}\left(\mu_{\rm F} e^{-L_{t,1}}\right) \right]$$
$$\times \left[\Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1} + \vec{k}_{t,s_1}|\Big) - \Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1}|\Big) \right] \Bigg\}, \qquad (38)$$

$$\sigma_{\rm clust}^{\rm NNLL}(p_t^{\rm J,v}, p_t^{\rm H,v}) = \int_0^{\infty} \frac{dk_{t,1}}{k_{t,1}} \frac{d\phi_1}{2\pi} \int d\mathcal{Z}\, e^{-R_{\rm NLL}(L_{t,1})} \mathcal{L}_{\rm NLL}\left(\mu_{\rm F} e^{-L_{t,1}}\right) 8\, C_A^2 \frac{\alpha_s^2}{\pi^2} \frac{L_{t,1}}{(1-2\beta_0 \alpha_s L_{t,1})^2} \Theta\left(p_t^{\rm J,v} - \max_{i>1}\{k_{t,i}\}\right)$$
$$\times \Bigg\{ \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{d\phi_{s_1}}{2\pi} \int_{-\infty}^{\infty} d\Delta\eta_{1s_1}\, J_{1s_1}(R) \left[\Theta\Big(p_t^{\rm J,v} - |\vec{k}_{t,1} + \vec{k}_{t,s_1}|\Big) - \Theta\Big(p_t^{\rm J,v} - k_{t,1}\Big) \right] \Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1} + \vec{k}_{t,s_1}|\Big)$$
$$+ \frac{1}{2!} \hat{R}'(k_{t,1}) \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{dk_{t,s_2}}{k_{t,s_2}} \frac{d\phi_{s_1}}{2\pi} \frac{d\phi_{s_2}}{2\pi} \int_{-\infty}^{\infty} d\Delta\eta_{s_1 s_2}\, J_{s_1 s_2}(R) \left[\Theta\Big(p_t^{\rm J,v} - |\vec{k}_{t,s_1} + \vec{k}_{t,s_2}|\Big) - \Theta\Big(p_t^{\rm J,v} - \max\{k_{t,s_1}, k_{t,s_2}\}\Big) \right]$$
$$\times\, \Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1} + \vec{k}_{t,s_1} + \vec{k}_{t,s_2}|\Big)\, \Theta\left(p_t^{\rm J,v} - k_{t,1}\right) \Bigg\}, \qquad (42)$$

$$\sigma_{\rm correl}^{\rm NNLL}(p_t^{\rm J,v}, p_t^{\rm H,v}) = \int_0^{\infty} \frac{dk_{t,1}}{k_{t,1}} \frac{d\phi_1}{2\pi} \int d\mathcal{Z}\, e^{-R_{\rm NLL}(L_{t,1})} \mathcal{L}_{\rm NLL}\left(\mu_{\rm F} e^{-L_{t,1}}\right) 8\, C_A^2 \frac{\alpha_s^2}{\pi^2} \frac{L_{t,1}}{(1-2\beta_0 \alpha_s L_{t,1})^2} \Theta\left(p_t^{\rm J,v} - \max_{i>1}\{k_{t,i}\}\right)$$
$$\times \Bigg\{ \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{d\phi_{s_1}}{2\pi} \int_{-\infty}^{\infty} d\Delta\eta_{1s_1}\, \mathcal{C}\left(\Delta\eta_{1s_1}, \Delta\phi_{1s_1}, \frac{k_{t,1}}{k_{t,s_1}}\right) \left(1 - J_{1s_1}(R)\right)$$
$$\times \left[\Theta\Big(p_t^{\rm J,v} - k_{t,1}\Big) - \Theta\Big(p_t^{\rm J,v} - |\vec{k}_{t,1} + \vec{k}_{t,s_1}|\Big) \right] \Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1} + \vec{k}_{t,s_1}|\Big)$$
$$+ \frac{1}{2!} \hat{R}'(k_{t,1}) \int_0^{k_{t,1}} \frac{dk_{t,s_1}}{k_{t,s_1}} \frac{dk_{t,s_2}}{k_{t,s_2}} \frac{d\phi_{s_1}}{2\pi} \frac{d\phi_{s_2}}{2\pi} \int_{-\infty}^{\infty} d\Delta\eta_{s_1 s_2}\, \mathcal{C}\left(\Delta\eta_{s_1 s_2}, \Delta\phi_{s_1 s_2}, \frac{k_{t,s_2}}{k_{t,s_1}}\right) \left(1 - J_{s_1 s_2}(R)\right) \Theta\left(p_t^{\rm J,v} - k_{t,1}\right)$$
$$\times \left[\Theta\Big(p_t^{\rm J,v} - \max\{k_{t,s_1}, k_{t,s_2}\}\Big) - \Theta\Big(p_t^{\rm J,v} - |\vec{k}_{t,s_1} + \vec{k}_{t,s_2}|\Big) \right] \Theta\Big(p_t^{\rm H,v} - |\vec{k}_{t,1} + \cdots + \vec{k}_{t,n+1} + \vec{k}_{t,s_1} + \vec{k}_{t,s_2}|\Big) \Bigg\}. \qquad (43)$$