Lie groups in Quantum Field Theory

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A person who is tired of group theory is tired of life
Sidney Coleman

1 Lie groups

Lie groups are a fundamental concept in physics, and in field theory in particular. This is because the most important symmetries which appear in field theory are continuos symmetries. We collect here some useful definitions and some basic concepts. First of all, let us remember what is a group.

Definition 1 (Group). *A group $\mathscr{G}$ is a set, together with a map $* : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(g_1, g_2) \to g_1 * g_2 \in \mathscr{G}$ for $g_1,\ g_2 \in \mathscr{G}$. The map $*$, called group muliplication, satisfies the following properties:*

(i) Existence of the identity element: $\exists\, e \in \mathscr{G}$ s.t. $e * g = g * e = g\ \forall g \in \mathscr{G}$.

(ii) Associativity: $g * (h * k) = (g * h) * k\ \forall g, h, k \in \mathscr{G}$.

(iii) Existence of the inverse: $\exists\, g^{-1} \in \mathscr{G}$ s.t. $g^{-1} * g = g * g^{-1} = e\ \forall g \in \mathscr{G}$.

A group is abelian if the group operation is commutative, i.e. $g * h = h * g\ \forall g, h \in \mathscr{G}$.

A Lie group is a group whose elements depend in a continuos and differentiable way on a set of parameters. More precisely, a Lie group is a smooth manifold with the algebraic structure of a group. We will now introduce some definitions necessary to introduce the concepts of differentiable and smooth manifold.

Differentiable manifolds play a central role in mathematics and physics. The notion of manifold extends the notion of (hyper)surface in $\mathbb{R}^n$, thus enabling to deal with very general spaces in a particularly versatile way. Essentially, a manifold is a set made up of pieces which locally looks like open subsets of $\mathbb{R}^n$ and which can be patched together smoothly. More precisely, we can give the following definition:

Definition 2 (Manifold). *A manifold is a Hausdorff space*[1] *$\mathscr{M}$ for which every point $x \in \mathscr{M}$ has a neighbourhood homeomorphic to Euclidean space $\mathbb{R}^n$.*

We would like now to equip the manifold with a differentiable structure. We call a local coordinate chart, or simply chart, a homeomorphism[2] $\varphi : U \to V$ of an open set $U \subseteq \mathscr{M}$ onto an open set

[1] A topological space X is Hausdorff if for any $x, y \in X,\ x \neq y$, there exist open sets U containing x and V containing y such that $U \cap V = \varnothing$.

[2] A homeomorphism is a bijective continuous map with continuous inverse

$V \subseteq \mathbb{R}^n$. A C^∞ differentiable structure is a collection of charts which are C^∞-compatible, in the sense defined below. Let us consider a topological space $\mathscr{M}$. We define a C^∞-differentiable structure as

Definition 3 (**Differentiable structure**). *A C^∞ differentiable structure, or smooth structure, on $\mathscr{M}$ is a collection of coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ such that*

(i) $\mathscr{M} = \cup_\alpha U_\alpha$*;*

(ii) any two charts are 'compatible': for every α, β the change of local coordinates $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth (C^∞) map on its domain of definition, i.e. on $\varphi_\alpha(U_\beta \cap U_\alpha) \subseteq \mathbb{R}^n$.

(iii) the collection of charts φ_α, called atlas, is maximal with respect to the property (ii): if a chart φ of $\mathscr{M}$ is compatible with all φ_α then φ is included in the collection.

Isomorphisms on smooth manifolds are called diffeomorphisms. A diffeomorphisms a bijective smooth map with a smooth inverse; diffeomorphisms are to differential manifolds as homeomorphisms are to topologies, and as isomorphism are to vector spaces. We are now ready to define a smooth manifold:

Definition 4 (**Smooth manifold**). *A topological space equipped with a C^∞ differential structure is called a smooth manifold.*

After this brief excursus on differential geometry, we can now define a Lie group as

Definition 5 (**Lie group**). *A Lie group is a finite dimensional smooth manifold $\mathscr{G}$ together with a group structure on $\mathscr{G}$, such that the multiplication $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ and $g \mapsto g^{-1} : \mathscr{G} \to \mathscr{G}$ are smooth maps.*

We observe that the general linear group $\mathrm{GL}_n(\mathbb{R})$

$$\mathrm{GL}_n(\mathbb{R}) \equiv \{A \in \mathrm{M}_n(\mathbb{R}) \mid \det A \neq 0\}, \tag{1.1}$$

where $\mathrm{M}_n(\mathbb{R})$ is the set of $n \times n$ matrices, is a Lie group.

An important concept to introduce is the concept of representations, which will let us describe abstract groups in terms of linear operators living in a vector space: in particular, in terms of matrices. A linear representation R of a Lie group $\mathscr{G}$ is an operation that assigns to an abstract element $g \in \mathscr{G}$ a linear operator defined in a vector space

$$g \mapsto D_R(g) \tag{1.2}$$

which maps the identity in the identity operator, $D_R(e) = 1$, and preserves the group structure, $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$. Technically speaking, a representation is a smooth group homomorphism $R : \mathscr{G} \to \mathscr{H}$, where $\mathscr{H} = \mathrm{GL}_n(\mathbb{R})$ or $\mathscr{H} = \mathrm{GL}_n(\mathbb{C})$ (or a Lie subgroup of these Lie groups). The dimension of the representation is defined as the the dimension n of the vector space, which is usually called base space. We will usually consider matrix representations, *i.e.* we will represent a generic element g of the group with a $n \times n$ matrix $(D_R(g))^i{}_j$. Writing a generic element of the base space $(v^1, \ldots, v^n)$, a group element g therefore induces a transformation of the vector space

$$v^i \to (D_R(g))^i{}_j v^j. \tag{1.3}$$

The abstract group element g acquires in this way a physical meaning, since we can now interprete g as a transformation on a certain space. For instance, taking as group SO(3) and as base space the spatial vectors $\vec{v}$, an element $g \in \mathrm{SO}(3)$ can be interpreted physically as a rotation in three-dimensional space.

We will be interested in irreducible representations. A representation is called reducible if it has an invariant subspace. That means the action of $D_R(g)$ on a vector of the invariant subspace gives another vector of that subspace. Representations with no invariant subspaces are called irreducible. We say that a representation is completely reducible if we can find a basis such that the matrices $D_R((g)$ are block-diagonal for all elements of g. Completely reducible representations can therefore be written as the direct sum of irreducible representations, and that is why irreducible representation are particularly interesting. We finally call the representations R and R' equivalent if we can find a matrix S, independent of g, such that $D_R(g) = S^{-1} D_{R'}(g) S$ for all g. Equivalent representation correspond to a change of basis in the vector space.

We observe that when we change the representation the explicit form and the dimension of the matrices will change as well. Therefore we might wonder whether there exist some properties of the Lie group which do not depend of the representation. To answer to this question, we introduce the concept of Lie algebra.

Lie algebras are an essential tool in studying Lie groups. The advantage of dealing with Lie algebras is twofold: firstly, Lie algebras are simpler than Lie groups since they are vector space. Therefore we can understand a lot about Lie algebras just by algebraic calculations. Secondly, Lie algebras of a Lie group contains much information about the Lie groups: therefore we can understand many properties of a Lie group by considering its (easier) Lie algebra.

We briefly recall the definition of tangent vector and tangent space:

Definition 6 (Tangent space). *Let $\mathscr{F}(\mathscr{M})$ be the set of the smooth functions $f : \mathscr{M} \mapsto \mathbb{R}$ on the manifold $\mathscr{M}$. A tangent vector to a manifold $\mathscr{M}$ at a point $p \in \mathscr{M}$ is a map*

$$v : \mathscr{F}(\mathscr{M}) \mapsto \mathbb{R}$$

with the following properties:

(i) $v(af + bg) = av(f) + bv(g)$

(ii) $v(fg) = v(f)g(p) + f(p)v(g)$

for $a, b \in \mathbb{R}$ and $f, g \in \mathscr{F}(\mathscr{M})$. The set of the tangent vectors at a given point p form the tangent space denoted $T_p\mathscr{M}$.

One can easily check that $T_p\mathscr{M}$ is a vector space.

The Lie algebra $\mathfrak{g}$ of a Lie group $\mathscr{G}$ is the tangent space to $\mathscr{G}$ at the identity $e \in \mathscr{G}$. $\mathfrak{g}$ is a vector space of dimension $n = \dim\mathscr{G}$, with an algebraic structure called Lie bracket. More precisely, we can define a Lie algebra as

Definition 7 (Lie algebra). *Let V be a vector space and $[,] : V \times V \to V$ a map with the following properties:*

(i) $[X, Y] = -[Y, X]$

(ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacobi identity)*

Then $(V,[,])$ *is a Lie algebra.*

Let us now consider a basis of $\mathfrak{g}$ $(X_1,\ldots,X_n)$. Since $[X_a,X_b]\in\mathfrak{g}$, we can now expand the Lie bracket in terms of X_i:

$$[X^a,X^b]=f^{ab}_{\ c}X^c. \tag{1.4}$$

Lie algebras (and Lie groups) are uniquely defined by their structure constants. $f_{ab}^{\ \ c}$ are called structure constants of $\mathcal{G}$ and, in a sense, determine a Lie group completely.

The link between a Lie group and its Lie algebra is provided by the exponential map. The notion of exponential map can be introduced straightforwardly in the case of matrix Lie groups. We have then

Definition 8 (Exponential map). *If* $\mathcal{G}$ *is a matrix Lie group with Lie algebra* $\mathfrak{g}$*, then the exponential map for* $\mathcal{G}$ *is the map*

$$\exp:\mathfrak{g}\to\mathcal{G}$$

where, if X *is an* $n\times n$ *matrix, we define the exponential of* X *by the usual power series*

$$e^X=\sum_{k=0}^{\infty}\frac{X^k}{k!},$$

where X^0 *is defined to be the identity matrix 1 and where* X^k *is the repeated matrix product of* X *with itself.*

In case of a matrix Lie group, then the Lie algebra of $\mathcal{G}$ can be identified with the set of all matrices X such that $e^{tX}\in\mathcal{G}$ for all real numbers t. Physicist are usually accustomed to the map e^{itX}, and in this notes we will stick to this convention. Keep in mind however that you might find expressions which differ by a factor of i depending on the literature.

We observe that not all the informations about the group can be obtained when one moves from the algebra to the group. In particular, all the informations about the elements of the group which are not connected with the identity are lost.

By restricting a representation D_R of $\mathcal{G}$ to elements close to the identity e, we can now obtain the notion of a representation d_R of the Lie algebra $\mathfrak{g}$.

Definition 9 (Representation of a Lie algebra). *A representation* d_R *of* $\mathfrak{g}$ *acting on a vector space* V *is a linear action* $v\to d_R(X)v$*, with* $X\in\mathfrak{g}$ *and* $v\in V$*, satisfying*

$$d_R([X,Y])=d_R(X)d_R(Y)-d_R(Y)d_R(X)=[d_R(X),d_R(Y)]$$

As with the case of groups, the dimension of the representation is n*, where* n *is the dimension of the vector space* V*.*

It can be proved that a representation of the group D_R induces naturally a representation of the algebra d_R. Thanks to the exponential map, we can obtain a representation of the group D_R which holds locally (*i.e.* in a neighbourhood of the identity): $D_R(\exp X)=\exp(d_R(X))$. In physics it is often easier to use representations of the algebra rather than the corresponding representations of the group.

Once we have chosen a particular representation, a generic group element connected with the identity can be represented as

$$D_R(g)=e^{i\vartheta_a T^a_R}, \tag{1.5}$$

where

$$T_R^a \equiv -i \frac{\partial D_R}{\partial \vartheta_a}\Big|_{\vartheta=0} \tag{1.6}$$

are called generators of the algebra (and usually, with an abuse of notation, of the group itself). We can understand now how the physicist prefer the convention with i at the exponent: if in the representation R the generators are hermitian, it follows that $D_R(g)$ are unitary. In this case R is a unitary representation.

We have seen therefore that the Lie algebra, with its bracket structure, determines the group structure of $\mathscr{G}$ near e. The problem of finding all matrix representations of a Lie algebra amounts to the algebraic problem of finding all possible matrix solutions to the equation

$$[T_R^a, T_R^b] = f^{ab}_{c} T^c. \tag{1.7}$$

The task is very simple for abelian groups, since for an abelian Lie group the structure constants vanish. In particular, any n-dimensional abelian Lie algebra is isomorphic to the direct sum of n one-dimensional abelian Lie algebras. In other words, all irreducible representations of abelian groups are one-dimensional. The non-trivial part of the representation theory of Lie algebras is related to the non-abelian structure.