So far, we've worked in the Schrödinger picture where states depend on time but operators don't. We can use $U(t)$ to pass to the Heisenberg picture where t-dependence is shifted from states to operators:

	Schrödinger	Heisenberg
states	$\|\psi(t)\rangle_S$	$\|\psi\rangle_H = e^{itH/\hbar}\|\psi(t)\rangle_S = \|\psi(0)\rangle$
op.s	A_S	$A_H(t) = e^{itH/\hbar} A_S e^{-itH/\hbar}$

Because transformation is unitary, ${}_S\langle\varphi(t)|A_S|\psi(t)\rangle_S = {}_H\langle\varphi|A_H(t)|\psi\rangle_H$, and all physical predictions are the same in either picture.

NB: $H_H = H_S = H$.

H-picture makes QM look more like classical mechanics (pos^n, momentum etc evolving in time). To specify dynamics we need an eq^n to tell us how op.s evolve in time (In S-picture, the SE tells us how states evolve).

$$\frac{d}{dt}A_H(t) = \frac{d}{dt}\left(e^{iHt/\hbar}A_S e^{-iHt/\hbar}\right) = \frac{iH}{\hbar}\overbrace{e^{iHt/\hbar}A_S e^{-iHt/\hbar}}^{A_H} - e^{iHt/\hbar}A_S e^{-iHt/\hbar}\frac{iH}{\hbar}$$
$$= \frac{i}{\hbar}[H, A_H(t)]$$

or $\quad i\hbar\frac{d}{dt}A_H(t) = [A_H(t), H]$ — Heisenberg eq^n of motion.

eg part^c in 1d $\hat{x}(t), \hat{p}(t)$ in H-picture (drop H subscripts), consider $H = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$$\frac{d}{dt}\hat{x}(t) = \frac{1}{i\hbar}[\hat{x}(t), H] \quad \left(= \frac{1}{2mi\hbar}(\hat{x}\hat{p}^2 - \hat{p}^2\hat{x}) = \frac{2i\hbar\hat{p}}{2mi\hbar}\right) = \frac{\hat{p}(t)}{m}$$

$$\frac{d}{dt}\hat{p}(t) = \frac{1}{i\hbar}[\hat{p}(t), H] \quad \begin{pmatrix}\text{action on state to RHS} \\ (\hat{p}(\hat{V}\psi) - \hat{V}\hat{p}\psi) = \hat{p}\hat{V}\psi - \hat{V}\hat{p}\psi \ldots\end{pmatrix} = -V'(\hat{x}(t))$$

Take exp. values in any state $|\psi\rangle$ – indep. of t in H-picture –: $\frac{d\langle\hat{x}\rangle}{dt} = \frac{1}{m}\langle\hat{p}\rangle$ and $\frac{d}{dt}\langle\hat{p}\rangle = -\langle V'(\hat{x})\rangle$

ie Ehrenfest's theorem, true in all pictures (since physical observables).

For some potentials, we can solve Heisenberg eq^{n}s:

• $V=0$, free part^c: $\quad \frac{d}{dt}\hat{p}(t) = 0 \Rightarrow \hat{p}(t) = \hat{p}(0)$ const. operator

$$\frac{d}{dt}\hat{x}(t) = \frac{1}{m}\hat{p}(0) \Rightarrow \hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)}{m}t$$

(sol^n just like in classical dynamics but with appearance of const. op.s.)

• $V = \frac{1}{2}m\omega^2x^2$, oscillator

$$\left.\begin{aligned}\frac{d}{dt}\hat{x}(t) &= \frac{\hat{p}(t)}{m} \\ \frac{d}{dt}\hat{p}(t) &= -\underbrace{m\omega^2\hat{x}(t)}_{V'(\hat{x})}\end{aligned}\right\} \Rightarrow \quad \frac{d^2}{dt^2}\hat{x} + \omega^2\hat{x} = 0, \quad \frac{d^2}{dt^2}\hat{p} + \omega^2\hat{p} = 0.$$

(from above, $\frac{d^2}{dt^2}$ ←)

Sol^n $\quad \hat{x}(t) = \hat{x}(0)\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t \quad$ (since $\dot{\hat{x}}(0) = \frac{\hat{p}(0)}{m}$), but can convert to $a, a^\dagger$ from

$\hat{p}(t) = \hat{p}(0)\cos\omega t - m\omega\hat{x}(0)\sin\omega t$ — Schrödinger picture op.s $\hat{x}(0)$ and $\hat{p}(0)$:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right), \quad \hat{p}(t) = \sqrt{\frac{\hbar\omega m}{2}}\frac{1}{i}\left(a e^{-i\omega t} - a^\dagger e^{i\omega t}\right)$$

(since $\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\cos\omega t + \sqrt{\frac{\hbar}{2m\omega}}\,i(a^\dagger - a)\sin\omega t$ etc.)

* Non-examinable * (c) Canonical Quantisation (The final step in Dirac's systematic approach to QM: we've seen how to incorporate pos^n, mom, wfs, S+H pictures within a single logical framework. But: how do we pass from a general classical system to its quantum version? In particular, what are the fundamental quantum mechanical comm. relations between them – eg$^{\text{lin}}$ $[\hat{x}, \hat{p}] = i\hbar$?)

Any classical system can be described by a set of generalised positions $x_i(t)$ and momenta $p_i(t)$ with $1 \le i \le N$ (may include angles + angular momenta in addition to pos^{n}s + momenta) and a Hamiltonian $H(x_i, p_i)$. The Poisson bracket of any f^{n}s $f(x_i, p_i)$ and $g(x_i, p_i)$ is defined to be $\{f, g\} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}$, a new f^n of x_i and p_i's. (x_i and p_i coords in phase-space and Poisson bracket is a symplectic structure). In particular, $\{x_i, p_j\} = \delta_{ij}$ $\left(= \sum_k \frac{\partial x_i}{\partial x_k}\frac{\partial p_j}{\partial p_k}\right)$

Properties of the PB which are easily checked include antisymmetry, bilinearity and Jacobi identity. Dynamics in this formulation is given by Hamilton's eq^n

$$\frac{df}{dt} = \{f, H\} \quad \text{for any } f(x_i, p_i).$$

In canonical quantisation define quantum theory by

classical f^{n}s $f, g \longrightarrow$ quantum ops $\hat{f}, \hat{g}$