The space V is complete: (assuming appropriate sequences / series converge).
A complete inner product space like this is a 'Hilbert space' - often V is called that in QM
(V can be finite or ∞-dimensional)

An operator Q is a linear map on states $(V \to V)$ $\quad |\psi\rangle \mapsto Q|\psi\rangle$
ie $Q(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha Q|\psi\rangle + \beta Q|\varphi\rangle$.
Some op. can be regarded as acting 'to the left' on dual states $(V^\dagger \to V^\dagger)$.
$\langle\varphi| \longmapsto \langle\varphi|Q$,
where the RHS is defined by knowing $(\langle\varphi|Q)|\psi\rangle = \langle\varphi|(Q|\psi\rangle)$ $\forall |\psi\rangle$.
or just write $\langle\varphi|Q|\psi\rangle$.

$\forall Q$, hermitian conjugate or adjoint is an op $Q^\dagger$, defined by $\langle\varphi|Q^\dagger = (Q|\varphi\rangle)^\dagger$ or equivalently $\langle\varphi|Q^\dagger|\psi\rangle = (Q|\varphi\rangle)^\dagger|\psi\rangle = \langle\psi|Q|\varphi\rangle^*$
$\Rightarrow$ for any op.s A, B: $\quad (\alpha A + \beta B)^\dagger = \alpha^* A^\dagger + \beta^* B^\dagger$, $\quad (AB)^\dagger = B^\dagger A^\dagger$.

For any op. Q, $|\psi\rangle^{(\neq 0)}$ is an eigenstate with e'value λ if $Q|\psi\rangle = \lambda|\psi\rangle$ (take †) $\Rightarrow \langle\psi|Q^\dagger = \lambda^*\langle\psi|$
For general Q, λ can be complex. Commutators take on ~~explicit~~ particular importance:
$[A,B] = AB - BA = -[B,A]$.

1b Observables + Measurements

An op. Q is hermitian or self-adjoint if $Q^\dagger = Q$: called "observables" - correspond to physical, measurable quantities eg position, momentum, energy, angular momentum.
Key results
(i) All eigenvalues are $\mathbb{R}$
(ii) E'states with distinct e'vals are orthogonal
(iii) Any state can be expanded in terms of (ie written as a linear combination of) e'states.

Prove (i), (ii) + assume (iii):
(i) $Q|\psi\rangle = \lambda|\psi\rangle \Rightarrow \langle\psi|Q^\dagger = \lambda^*\langle\psi|$ $\Rightarrow \langle\psi|Q|\psi\rangle = \lambda\langle\psi|\psi\rangle = \lambda^*\langle\psi|\psi\rangle$
$= \langle\psi|Q$ if Q hermitian. and $\langle\psi|\psi\rangle \neq 0$ so $\lambda = \lambda^*$.
(ii) Let $|n\rangle$ be e'states of Q, evals $\lambda = q_n$ real (n - discrete label, poss. ∞)
$Q|n\rangle = q_n|n\rangle$ and $Q|m\rangle = q_m|m\rangle$ or $\langle m|Q = q_m\langle m|$.
$\Rightarrow \langle m|Q|n\rangle = q_n\langle m|n\rangle = q_m\langle m|n\rangle$. so $q_n \neq q_m \Rightarrow \langle m|n\rangle = 0$.
Combining these 3 properties we have:
* For any observable Q $\exists$ an orthonormal basis of e'states $\{|n\rangle\}$ for the space of states V with $Q|n\rangle = q_n|n\rangle$, $\langle m|n\rangle = \delta_{mn}$
This means that a general state can be expanded $|\psi\rangle = \sum_n \alpha_n|n\rangle$ where $\alpha_n = \langle n|\psi\rangle$

For the state to be properly normalised $\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = 1$. $\Leftrightarrow$ $\left(\sum_m \alpha_m^*\langle m|\right)\left(\sum_n \alpha_n|n\rangle\right) = \sum_n |\alpha_n|^2 = 1$.

There may be several states with the same e'val λ. Define the e'space for a given eval
$V_\lambda = \{|\psi\rangle : Q|\psi\rangle = \lambda|\psi\rangle\}$ which has basis $\{|n\rangle : q_n = \lambda\}$

The "degeneracy" of λ is the # of states in this basis, or $\dim V_\lambda$.
λ is "non-degenerate" if $\dim V_\lambda = 1$.

[Passing from our three key results to the conclusion (*) is achieved by choosing an orthonormal basis for each V_λ. (ii) ensures that these spaces are mutually orthogonal, (iii) $\Rightarrow$ sum of all eigenspaces is V, the entire space of states.]

A projection operator P_λ onto any e'space $(V \to V_\lambda)$ can be defined by its action on basis states $P_\lambda|n\rangle = \begin{cases} |n\rangle & \text{if } q_n = \lambda \\ 0 & \text{otherwise} \end{cases}$: $\sum_\lambda P_\lambda = 1$ $\quad P_\lambda^2 = P_\lambda$
$\lambda \neq \mu \Rightarrow P_\lambda P_\mu = P_\mu P_\lambda = 0$.

Consider a measurement of Q when the system is in a state $|\psi\rangle$ immediately before.
Then,
- The result is an e'value λ
- This is obtained with probability $p(\lambda) = \|P_\lambda|\psi\rangle\|^2 = \sum_{n: q_n = \lambda} |\alpha_n|^2$.