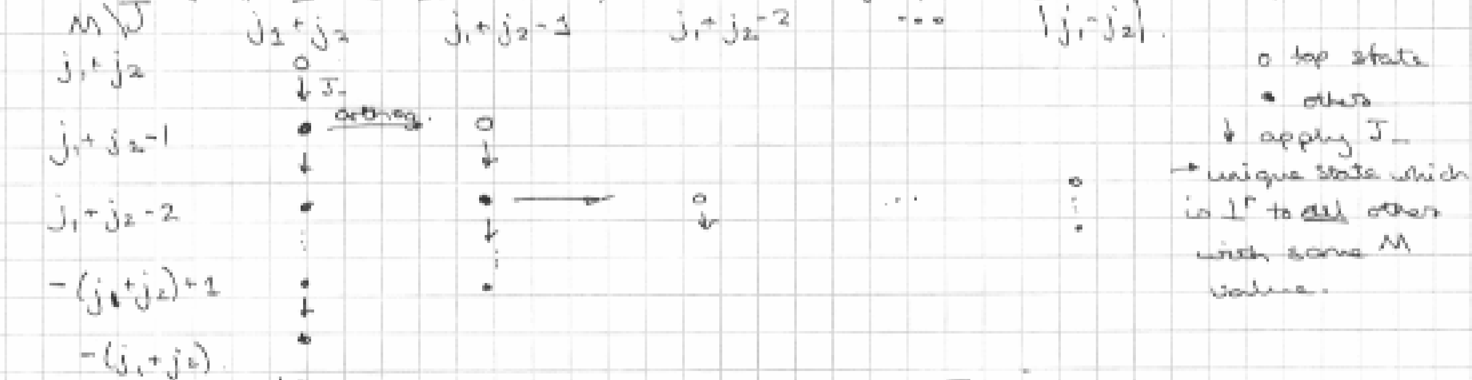
Act on it with J_3, eval j_1+j_2-1. (*) Call it $|\alpha\rangle$, then $\langle\alpha|j_1+j_2, j_1+j_2-1\rangle=0$
$\Leftrightarrow$ $\langle\alpha|J_-|j_1+j_2, j_1+j_2\rangle=0$, hence $\langle\alpha|J_-=0 \Leftrightarrow J_+|\alpha\rangle=0$.

$$|\alpha\rangle = |\underbrace{j_1+j_2-1}_{J}, \underbrace{j_1+j_2-1}_{M}\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1\rangle|j_2-1\rangle - \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1-1\rangle|j_2\rangle.$$

Now we proceed as before: generate multiplet with $J=j_1+j_2-1$ by applying J_-; look for a possible new top state at level $M=j_1+j_2-2$ and so on until all states are exhausted. The general pattern may be depicted as follows

[NB $\underline{J}^2 = \underline{J}_1^2+\underline{J}_2^2+2\underline{J}_1\cdot\underline{J}_2$, could re-write in terms of J_+, J_- and J_3 to define $\underline{J}^2$ ... see question sheets].



The top states $|\psi\rangle$ can be checked to be annihilated by J_+ directly, but this is guaranteed by $|\psi\rangle \perp J_-|\varphi\rangle$ $\forall\,|\varphi\rangle$ with some M value.
$\Leftrightarrow$ $\langle\varphi|J_+|\psi\rangle=0$ $\forall\,|\varphi\rangle$ with correct M value
$\Leftrightarrow$ $J_+|\psi\rangle=0$.

(*) $J_3|\alpha\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\left(j_1|j_1\rangle|j_2-1\rangle+(j_2-1)|j_1\rangle|j_2-1\rangle\right) - \sqrt{\frac{j_2}{j_1+j_2}}\left((j_1-1)|j_1-1\rangle|j_2\rangle+j_2|j_1-1\rangle|j_2\rangle\right)$ $= (j_1+j_2-1)|\alpha\rangle$

Whole process stops with $J=|j_1-j_2|$ by counting:

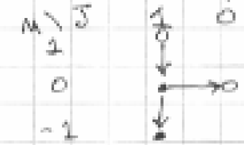
$$\sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1) = (2j_1+1)(2j_2+1)$$, hence counting is correct

$$\text{LHS} = (j_1+j_2-|j_1-j_2|)\cdot\frac{(2|j_1-j_2|+1+2(j_1+j_2)+1)}{2} = (j_1+j_2-|j_1-j_2|+1)\cdot(|j_1-j_2|+j_1+j_2+1) = (j_1+j_2+1)^2-(j_1-j_2)^2$$

$$= (j_1^2+2j_1j_2+2(j_1+j_2)+1)-(j_1^2-2j_1j_2+j_2^2) = 4j_1j_2+2(j_1+j_2)+1 = (2j_1+1)(2j_2+1).$$ ✓

(i) We've now found answers to original questions. $j_1+j_2 \geq J \geq |j_1-j_2|$ in integer steps
eg $j_1=j_2=\frac{1}{2}$ $\Rightarrow$ $J=1,0$ $\quad$ $j_1=1, j_2=\frac{1}{2}$ $\Rightarrow$ $J=\frac{3}{2},\frac{1}{2}$ $\quad$ $j_1=j_2=1$ $\Rightarrow$ $J=2,1,0$.
The range of $\underline{J}$ matches the bounds for addition of classical vectors $\underline{J}=\underline{J}^{(1)}+\underline{J}^{(2)}$ with vector lengths J, j_1 and j_2.

(ii) Clebsch-Gordan coefficients are found by explicit calculation of states for given j_1, j_2.
NB j_1, j_2 do not have to be different particles, eg take particle with $L=1$ and $S=\frac{1}{2}$, say, $\Rightarrow j_{tot}=\frac{3}{2},\frac{1}{2}$.
Work the simplest example. $j_1=j_2=\frac{1}{2}$, $|m_1\rangle|m_2\rangle$ $m_i=\pm\frac{1}{2}$, 4 states.
$|1\,1\rangle=|\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle$ is the top state with $M=1$. $\quad J_-=J_-^{(1)}+J_-^{(2)}$ gives
$\sqrt{2}\,|1\,0\rangle=|-\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle+|\tfrac{1}{2}\rangle|-\tfrac{1}{2}\rangle$ $\Rightarrow$ $2|1\,-1\rangle=|-\tfrac{1}{2}\rangle|-\tfrac{1}{2}\rangle+|-\tfrac{1}{2}\rangle|-\tfrac{1}{2}\rangle$ or $|1\,-1\rangle=|-\tfrac{1}{2}\rangle|-\tfrac{1}{2}\rangle$ completes $J=1$
then $|0\,0\rangle=\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\rangle|-\tfrac{1}{2}\rangle-|-\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle\right)$ is unique state with $M=0$ and $\perp^r$ to $|1\,0\rangle \Rightarrow$ it's $|0\,0\rangle$.



cf $\S$4: combinations of spin states $|\tfrac{1}{2}\rangle=|\uparrow\rangle$ and $|-\tfrac{1}{2}\rangle=|\downarrow\rangle$ we found previously by requiring definite symmetry
3 (triplet) $J=1$ states symm. spins
1 (singlet) $J=0$ state anti-symm. spins.

7. Transformations and Symmetries

(a) Introduction and an example. Given a unitary op. U we define a transformation of a quantum system to be either a map of states (only)
$|\psi\rangle \mapsto U|\psi\rangle$, $\langle\varphi| \longmapsto \langle\varphi|U^+$ or a map of op.s (only) $A \mapsto U^+AU$
These are equivalent, either way inner products of states $\langle\varphi|\psi\rangle$ unchanged but matrix elements change $\langle\varphi|A|\psi\rangle \mapsto \langle\varphi|U^+AU|\psi\rangle$
N.B. unlike change of picture, we change states or operators but not both. Such a transformation is a symmetry of the quantum system if $U^+HU=H$ or $[U,H]=0$
It follows that · in S-picture $|\psi(t)\rangle$ is sol^n of SE $\Rightarrow U|\psi(t)\rangle$ is too (since $UH|\psi\rangle=i\hbar U\frac{d\psi}{dt} \Rightarrow H(U\psi)=i\hbar\frac{d}{dt}(U\psi)$)
· in H-picture $A(t)$ is sol^n of HE $\Rightarrow U^+A(t)U$ is too.
$\left(\frac{dA}{dt}=\frac{i}{\hbar}[H,A] \Rightarrow \frac{d}{dt}(U^+AU)=\frac{i}{\hbar}(U^+HAU-U^+AHU)=\frac{i}{\hbar}\left[\overbrace{(U^+HU)}^{H}U^+AU-U^+AU\overbrace{(U^+HU)}^{H}\right]\right).$
Consider a group G and transformations of a QM system $U(g)$ for each $g\in G$ with
$U(g_1)U(g_2)=U(g_1g_2)$ $\quad$ $U(1_G)=1$ $\quad$ $U(g^{-1})=U(g)^{-1}=U(g)^+$