We usually write $\underline{J} = \underline{S}$ for spin; our analysis shows we must have $j = S$ either integral or ½-integral. Previously, we wrote $|s\rangle$ with $S \geq s \geq -S$ for spin states. Now, we write $|s\rangle = |j\ m\rangle$ with $j = S,\ m = s$.

Entire set of states $\{|j\,m\rangle\}$ for fixed j is often called an ∡r momentum representation. From our analysis above, we can choose states with

$$J_+ |j\,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\ |j\ m+1\rangle$$
$$J_- |j\,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\ |j\ m-1\rangle$$

– key relations between normalised states.

Whole multiplet is defined by taking top state $J_+|j\ j\rangle = 0$ (max J_3 e.val) and others $J_-^n |j\ j\rangle$ up to normalisation.

E.vals of $\underline{n}\cdot\underline{J}$ same as those of J_3, but e'states are linear combinations of those above (rotations).

(c) Matrix Representations.

Recall (§1d) that, given an orthonormal basis $\{|n\rangle\}$ we can regard states as column vectors $|\psi\rangle \leftrightarrow \alpha_n = \langle n|\psi\rangle$, and operators as matrices $A \leftrightarrow A_{mn} = \langle m|A|n\rangle$ with action of op.s on states as matrix multiplication – especially useful when space of states is finite-dimensional, eg ∡r mom. states $\{|j\,m\rangle\}$ with fixed j.

eg, with $j = 1$ (3-d state space)

$|1, 1\rangle \leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\quad |1, 0\rangle \leftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\quad |1\ {-1}\rangle \leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$

$J_3 \rightarrow \begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix}$ $\quad J_+ \leftrightarrow \hbar\sqrt{2}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ $\quad J_- \leftrightarrow \hbar\sqrt{2}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$

These follow from $J_3\,|1\ {\pm 1 \atop 0}\rangle = {\pm 1 \atop 0}\hbar\,|1\ {\pm 1 \atop 0}\rangle$ $\quad J_\pm |1\ {\mp 1}\rangle = \hbar\sqrt{2}\,|1\ 0\rangle$ $\quad J_\pm |1\ 0\rangle = \hbar\sqrt{2}\,|1, \pm 1\rangle$ as the only non-zero results.

Matrix rep.s are widely used for spin-½ (now use $\underline{S}$ rather than $\underline{J}$), with just 2 states:

$|½, ½\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |½\ {-½}\rangle \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$ $\quad$ ($|\uparrow\rangle, |\downarrow\rangle$ previously).

$S_3 \leftrightarrow \frac{1}{2}\hbar\begin{pmatrix}1&\\&-1\end{pmatrix}$ $\quad S_+ \leftrightarrow \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\quad S_- \leftrightarrow \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

Moreover, we write $S_i \leftrightarrow \frac{1}{2}\hbar\,\sigma_i$

where $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ are the '<u>Pauli matrices</u>'.

$\sigma_\pm = \sigma_1 \pm i\sigma_2 = \begin{pmatrix}0&2\\0&0\end{pmatrix}$, $\sigma_- = \begin{pmatrix}0&0\\2&0\end{pmatrix}$. The Pauli matrices are hermitian, traceless 2×2 obeying $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$ and $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$ (and cyclic combinations).

These properties are conveniently summarised by $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$. – (*)

NB// The anti-sym. part of (*) $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$ corresponds to fundamental comm. rel^n $[S_i, S_j] = i\hbar\,\varepsilon_{ijk}S_k$ but the remaining symmetric (anticomm.) part is special to spin ½.

The Pauli matrices are components of a vector $\underline{S} \leftrightarrow \frac{1}{2}\hbar\underline{\sigma}$, $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. If $\underline{a}, \underline{b}$ are constant vectors (or op.s which commute with $\underline{\sigma}$) then we can contract (*) with $a_i b_j$:

$$(\underline{a}\cdot\underline{\sigma})(\underline{b}\cdot\underline{\sigma}) = (\underline{a}\cdot\underline{b})\,1 + i(\underline{a}\wedge\underline{b})\cdot\underline{\sigma}.$$

As a special case, $(\underline{n}\cdot\underline{\sigma})^2 = 1$ where $\underline{n}$ is any unit vector. This is equiv. to $(\underline{n}\cdot\underline{S})^2 = \hbar^2/4$, ie e.vals of $\underline{n}\cdot\underline{S}$ are $\pm\hbar/2$ no matter what direction $\underline{n}$ points (direction spin is measured in).

NB// $\underline{\sigma}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 3$, cf $\underline{S}^2 = \hbar^2(\frac{1}{2})(\frac{1}{2}+1) = \frac{3}{4}\hbar^2$ on any $s = ½$ state (gives a last eg of matrix properties corresponding to known properties of operators).

(d) Some physical aspects of ∡r mom. & spin

(Our analysis of ∡r mom. states in 6b revealed the mathematical possibility of half-integral spin. As asserted earlier, this is realised in nature: previously stated that each part^c carries an internal space of states $|s\rangle$ with $-S \leq s \leq S$, now identify these with $|j = S, m = s\rangle$ ∡r mom. states.

The spin op.s obey $[S_i, \hat{x}_j] = [S_i, \hat{p}_j] = 0$ – <u>different</u> to comm. rel^{n}s for $\underline{L}$ (cf §6a). This is consistent with our earlier use of basis of states $|\underline{x}\rangle \otimes |s\rangle$ (fixed S) because $\underline{\hat{x}}, \underline{S}^2, S_3$ are actually a commuting set.

In general, a given particle has both orbital ∡r mom. $\underline{L}$ and spin ∡r mom. $\underline{S}$ giving total ∡r mom. $\underline{J} = \underline{L} + \underline{S}$. NB// $[S_i, L_j] = 0$ (since $\underline{L} \propto \underline{x}\wedge\underline{p}$) and hence $[L_i, L_j] = i\hbar\,\varepsilon_{ijk}L_k$ and $[S_i, S_j] = i\hbar\,\varepsilon_{ijk}S_k$ imply $[J_i, J_j] = i\hbar\,\varepsilon_{ijk}J_k$.

(How do we know the world works this way? – Results of many experiments – here, we just mention the theoretical ideas underlying a few of them. The key idea is how spin or ∡r mom. d.o.f. might enter into H – main eg for us is int^n with a background magnetic field.)